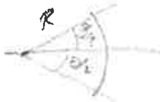
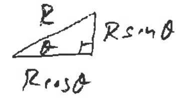
Arc of a circle, radius R, charge Q, subtending θ, evaluated at center

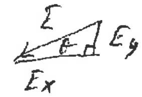


$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, ds}{R^2} \qquad \lambda = \frac{Q}{\theta R}$$



$$ds = R\, d\theta$$

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\, d\theta}{R^2} = \frac{\lambda}{4\pi\varepsilon_0 R}\, d\theta$$



$$dE_x = -\cos\theta\, dE = \frac{-\lambda}{4\pi\varepsilon_0 R}\cos\theta\, d\theta$$

$$E_x = \int dE_x = \frac{-\lambda}{4\pi\varepsilon_0 R}\int_{-\theta/2}^{+\theta/2}\cos\theta\, d\theta = \frac{-\lambda}{4\pi\varepsilon_0 R}\sin\theta\Big|_{\theta=-\theta/2}^{\theta=+\theta/2} = \frac{-\lambda}{4\pi\varepsilon_0 R}\left(\sin\frac{\theta}{2}+\sin\frac{\theta}{2}\right)$$

$$= \frac{-\lambda}{2\pi\varepsilon_0 R}\sin\left(\frac{\theta}{2}\right) = \frac{-Q}{2\pi\varepsilon_0 R^2}\frac{\sin(\theta/2)}{\theta}$$

$$E_y = \int dE_y = \frac{-\lambda}{4\pi\varepsilon_0 R}\int_{-\theta/2}^{\theta/2}\sin\theta\, d\theta = \frac{-\lambda}{4\pi\varepsilon_0 R}\left(-\cos\theta\Big|_{\theta=-\theta/2}^{\theta=+\theta/2}\right) = \frac{-\lambda}{4\pi\varepsilon_0 R}\left(-\cos\left(\frac{\theta}{2}\right)+\cos\left(\frac{\theta}{2}\right)\right) = 0$$

Some limiting cases. For $\theta = 2\pi$ we ought to get $E = 0$.

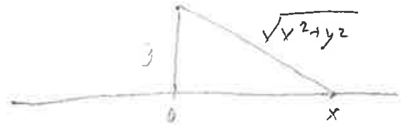
$$E_x = \frac{-Q}{2\pi\varepsilon_0 R^2}\frac{\sin(\pi)}{2\pi} = 0 \quad \text{because } \sin(\pi) = 0 \; ✓$$

For $\theta = 0$ we ought to get Colomb's law

$$E_x = \frac{-Q}{2\pi\varepsilon_0 R^2}\frac{\sin(\theta/2)}{\theta} \quad \text{use small angle approx}$$

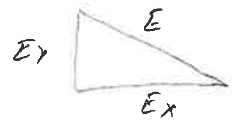
$$= \frac{-Q}{2\pi\varepsilon_0 R^2}\frac{\theta/2}{\theta} = \frac{-Q}{4\pi\varepsilon_0 R^2} \quad \text{Colomb's law} \; ✓$$

Line of Charge



$dq = \lambda\, dx$

$$dE = \frac{1}{4\pi\varepsilon_0}\ \frac{\lambda\, dx}{x^2+y^2}$$



$$dE_y = dE\ \frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{\lambda y}{4\pi\varepsilon_0}\ \frac{dx}{(x^2+y^2)^{3/2}}$$

$$E_y = \int E_y = \frac{\lambda y}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{dx}{(x^2+y^2)^{3/2}}$$

look up this integral
In the textbook it is integral 19 on page A-11

$$E_y = \frac{\lambda y}{4\pi\varepsilon_0}\ \frac{x}{y^2\sqrt{x^2+y^2}}\Bigg|_{x=-\infty}^{x=+\infty} = \frac{1}{y}\ \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{x}{\sqrt{x^2+y^2}}\Bigg|_{x=-\infty}^{x=+\infty}\right] = \frac{1}{y}\ \frac{\lambda}{4\pi\varepsilon_0}\left[1-(-1)\right]$$

$$= \frac{\lambda}{2\pi\varepsilon_0 y}$$ inverse-r dependence

Plane of Charge

$$dE = \frac{1}{4\pi\varepsilon_0}\ \frac{\sigma\, dy\, dx}{x^2+y^2+z^2} \qquad dE_z = \frac{z}{\sqrt{x^2+y^2+z^2}}\, dE = \frac{\sigma z}{4\pi\varepsilon_0}\ \frac{dy\, dx}{(x^2+y^2+z^2)^{3/2}}$$

Then $$E = \int dE = \frac{\sigma z}{4\pi\varepsilon_0}\int_x\int_y \frac{dy\, dx}{(x^2+y^2+z^2)^{3/2}} = \frac{\sigma z}{4\pi\varepsilon_0}\int_x \frac{y\ \ dx}{(x^2+z^2)\sqrt{x^2+y^2+z^2}}\Bigg|_{y=-\infty}^{y=+\infty}$$

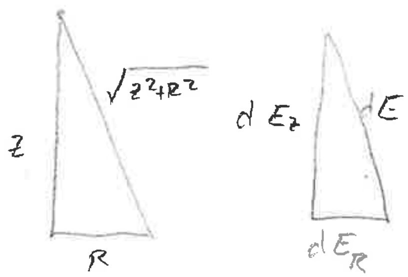
$$= \frac{\sigma z}{4\pi\varepsilon_0}\int_x \frac{dx}{(x^2+z^2)}\left[1-(-1)\right] = \frac{\sigma z}{2\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dx}{x^2+z^2} \overset{\text{integral table}}{=} \frac{\sigma z}{2\pi\varepsilon_0}\ \frac{1}{z}\, \operatorname{atan}\left(\frac{x}{z}\right)\Bigg|_{x=-\infty}^{x=+\infty}$$

$$= \frac{\sigma}{2\pi\varepsilon_0}\left[\operatorname{atan}(\infty) - \operatorname{atan}(-\infty)\right] = \frac{\sigma}{2\pi\varepsilon_0}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\sigma\pi}{2\pi\varepsilon_0} = \frac{\sigma}{2\varepsilon_0}$$

Ring of charge, evaluated at a field point on the principal axis

Make the z axis the principal axis, place the ring in the xy plane

$\lambda = \frac{Q}{2\pi R}$ $\quad dQ = \lambda R\, d\theta \quad dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda R\, d\theta}{z^2+R^2}$



so $dE_z = \frac{z}{\sqrt{z^2+R^2}}\, dE = \frac{\lambda R z}{4\pi\epsilon_0}\frac{d\theta}{(z^2+R^2)^{3/2}}$

$$E_z = \int dE_z = \frac{\lambda R z}{4\pi\epsilon_0 (z^2+R^2)^{3/2}}\int_0^{2\pi} d\theta = \frac{2\pi\lambda R}{4\pi\epsilon_0}\frac{z}{(z^2+R^2)^{3/2}} = \frac{Q}{4\pi\epsilon_0}\frac{z}{(z^2+R^2)^{3/2}}$$

check at $z=0$ and $z \gg R$ or $R=0$

at $z=0$ obviously $E_z = 0$

at $R=0$ we have $\frac{2\pi Q R}{4\pi\epsilon_0\, 2\pi R}\frac{z}{z^3} = \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}$, Coulomb's law

Disk of charge, field point on the principal axis

$\sigma = \frac{Q}{\pi R^2} \quad dQ = \sigma 2\pi r\, dr$

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{z\, dQ}{(z^2+R^2)^{3/2}} = \frac{z}{4\pi\epsilon_0}\frac{\sigma 2\pi r\, dr}{(z^2+r^2)^{3/2}}$$

$$E_z = \int E_z = \frac{2\pi\sigma z}{4\pi\epsilon_0}\int_0^R \frac{r\, dr}{(z^2+r^2)^{3/2}}$$

substitution $u = z^2+r^2$, $\frac{du}{dr} = 2r$ so $dr = \frac{du}{2r}$

$$E_z = \frac{2\pi\sigma z}{4\pi\epsilon_0}\int_0^R \frac{r\, du}{2r\, u^{3/2}} = \frac{2\pi\sigma z}{4\pi\epsilon_0}\int_0^R \frac{1}{2} u^{-3/2} du = \frac{2\pi\sigma z}{4\pi\epsilon_0}\left(-\tfrac{1}{2}\right)(2)\, u^{-1/2}\Big|_{r=0}^{r=R}$$

$$= \frac{2\pi\sigma z}{4\pi\epsilon_0}\left.\frac{-1}{\sqrt{z^2+r^2}}\right|_{r=0}^{r=R} = \frac{2\pi\sigma z}{4\pi\epsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right] = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$

check $R=0$ we have $\frac{Q}{2\pi R^2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) = \frac{Q}{2\pi\epsilon_0}\lim_{R\to 0}\frac{\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)}{R^2}$ ← use L'Hospital's Rule

but for an infinite plane, $R = \infty$ and $E_z = \frac{\sigma}{2\epsilon_0}(1-0) = \frac{\sigma}{2\epsilon_0}$ ✓

(over)

Check this form $\lim_{R \to 0} \frac{1}{R^2} \left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$

Take the derivative with respect to R

numerator is $1 - \frac{z}{\sqrt{z^2+R^2}} = u$ $\quad \frac{du}{dR} = 0 - \left[-\frac{1}{2}(z^2+R^2)^{-3/2} z \, 2R\right] = \frac{+Rz}{(z^2+R^2)^{3/2}}$

denominator is R^2 derivative is $2R$

Then $\lim_{R \to 0} \frac{\frac{Rz}{(z^2+R^2)^{3/2}}}{2R} = \lim_{R \to 0} \frac{z}{2(z^2+R^2)^{3/2}} = \frac{1}{2} \frac{z}{z^3} = \frac{1}{2z^2}$

Then the complete expression becomes $\frac{Q}{2\pi\varepsilon_0}\left(\frac{1}{2z^2}\right) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{z^2}$

And it works!