Worksheet 2: Trajectory Kinematics

Objectives

- Predict and analyze free-fall motion.
- Determine velocity and position for non-constant acceleration.
- Represent position, velocity and acceleration as vectors.

Summary

Free fall

Acceleration = $g = 9.8$ m/s^2 downward. So $y = y_0 + v_0 t \pm \frac{1}{2} g t^2$ and $v = v_0 \pm gt$.
(The ± depends on the sign of the downward direction.)

Non-constant acceleration

$v = v_0 + \int_0^t a\,dt;\ x = x_0 + \int_0^t v\,dt$

Position, velocity, and acceleration vectors

Position $\vec{r} = (x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$

Velocity $\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

Speed $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \left(\frac{dv_x}{dt}, \frac{dv_y}{dt}, \frac{dv_z}{dt}\right) = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right) = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

Component of $\vec{a}$ parallel to $\vec{v}$: $a_{\parallel}$ = rate of change of speed = dv/dt

Component of $\vec{a}$ perpendicular to $\vec{v}$: $a_{\perp}$. Affects direction of $\vec{v}$ only.

Problems

Remember: there is not room on this worksheet for your work. Use copious amounts of your own scratch paper!

1. Three identical steel balls are released at the same time from the same height above the ground. One is released with initial speed 0 m/s, one with initial speed v_0 upward, and one with initial speed v_0 downward. Once released, all are in free-fall until they hit the ground.

a. Draw a diagram of the initial situation. Show axis directions and the location of the origin.

b. Construct, for each ball, the kinematic equation giving height as a function of time.

c. At what time (how many seconds after their release) does each ball hit the ground? Since you have not been given any numbers, these will be three formulas.

d. Check the units for each formula in part c.

e. For the ball that was released with an initial upward speed and for the ball that was released with an initial downward speed, find the height as a function of time for the limiting case $v_0 = 0$ m/s.

f. Are these formulas found in part e what they should be?

g. Construct equations for the height differences between:

- The ball initially moving upward and the ball released from rest.
- The ball released from rest and the ball initially moving downward.

h. Find the maximum heights above the ground reached by:

- The ball initially moving upward.
- The ball initially moving downward.

Hint: at what *time* is the height at its maximum? What is the velocity then?

i. What is the physical meaning of the maximum height found for the downward-moving ball?

j. Substitute the height found in part h into the height equation for the ball released from rest and solve for t to find when this ball reaches that height. What do you find?

3. The x- and y-coordinates of a projectile in a free-fall trajectory typically vary with time as $x = ct$ and $y = dt^2$.

a. What must be the units of c and d?

b. Express the position vector $\vec{r}$ as a function of t using the basis vectors $\hat{\imath}$ and $\hat{\jmath}$.

c. Find the velocity components $v_x = dx/dt$ and $v_y = dy/dt$.

d. Express the velocity vector $\vec{v}$ as a function of t using the basis vectors $\hat{\imath}$ and $\hat{\jmath}$.

e. Find the speed $v = \sqrt{v_x^2 + v_x^2}$.

f. Find the acceleration components $a_x = dv_x/dt$ and $a_y = dv_y/dt$.

g. Express the acceleration vector $\vec{a}$ as a function of time t using the basis vectors $\hat{\imath}$ and $\hat{\jmath}$.

h. Find a, the magnitude of acceleration.

i. Find the rate of change of speed dv/dt. Recall that $\frac{d}{dx}\left(\sqrt{u}\right) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$.

j. Compare dv/dt to a. Should they be the same?

k. Find the parallel component of acceleration $a_{\parallel} = \frac{\vec{a} \cdot \vec{v}}{v}$.

l. Compare dv/dt to $a_{\parallel}$. Should they be the same?

2. A simple model of the acceleration of an object released from rest and falling through a viscous fluid is $a = -ge^{-bt}$, where b is a positive number indicating the viscosity of the medium, the upward direction is positive, and $v_0 = 0$ m/s.

a. Make a diagram of the starting conditions for such an object. Show axis directions and the location of the origin.

b. Sketch qualitative acceleration-time plots for small and large values of b.

c. Qualitatively sketch the velocity-time plot you expect from the acceleration. Is velocity upward or downward?

d. What is the expression for a as a function of t in the special case of $b = 0$?

e. Given your answer to part d, what should v be in the special case of $b = 0$?

f. Integrate the expression for acceleration to obtain an expression for velocity. Remember that $\int e^{ax}dx = e^{ax}/a$.

g. What is this expression for velocity in the special case of $b = 0$? To avoid getting expressions of 0/0 that are impossible to evaluate, find the limit of v as $b \to 0$. Use the Taylor expansion of the exponential function: $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$. Is the expression what it should be?

h. In the general case that $b > 0$, what is v when $t = \infty$?

i. Qualitatively sketch the position-time plot you expect from the velocity.

j. Integrate the expression for velocity to obtain the formula for position as a function of time. Is the formula consistent with your sketched prediction?

k. What is the position as a function of time in the special case of $b = 0$? Again use the Taylor expansion of the exponential function. Is this limiting formula what it should be?