Discussion 2 Solutions

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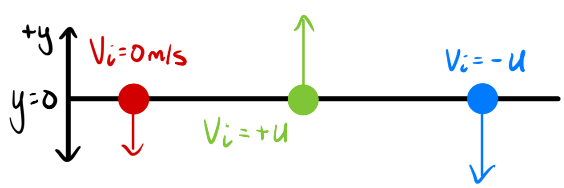
I. Three identical steel balls are released at the same time from the same height above the ground. One is released with initial speed 0 m/s, one with initial speed u upward, and one with initial speed u downward. Once released, all are in free-fall until they hit the ground.

A. Draw a diagram of the initial situation. Show axis directions and the location of the origin.



B. Construct, for each ball, the kinematic equation giving height as a function of time.

Since this situation only involves motion in the y direction (with the only acceleration being due to gravity, g) we will use the following equation for all three balls:

$$y_f = y_0 + v_{0,y}t - \frac{1}{2}gt^2. \tag{1}$$

We defined our origin to be at the starting point of the balls (this is allowed, but not required). Therefore, y_0 for all three balls will be zero. For the first ball, $v_{0,y}$ is also zero. Therefore, the equation of motion for the first ball, $y_{f,1}$ is:

$$y_{f,1} = 0 + 0 - \frac{1}{2}gt^2 = -\frac{1}{2}gt^2. \tag{2}$$

In other words, the only motion that this ball experiences is due to gravity. The second ball has an initial velocity of $v_{y,0} = +u$. Therefore, the equation of motion $y_{f,2}$ is then:

$$y_{f,2} = 0 + ut - \frac{1}{2}gt^2 = ut - \frac{1}{2}gt^2. \tag{3}$$

Finally, the third ball has an initial velocity of $v_{y,0} = -u$. Therefore, the equation of motion, $y_{f,3}$, is then:

$$y_{f,3} = 0 - ut - \frac{1}{2}gt^2 = -ut - \frac{1}{2}gt^2. \tag{4}$$

C. Construct equations for the height differences as functions of time between:

1. The ball initially moving upward and the ball released from rest.

The function for the height difference between the ball initially moving upward (ball 2) and the ball released from rest (ball 1) is the difference between the two equations of motion:

$$y_{f,2} - y_{f,1} = ut - \frac{1}{2}gt^2 - \left(-\frac{1}{2}gt^2\right) = ut - \frac{1}{2}gt^2 + \frac{1}{2}gt^2 = ut. \tag{5}$$

2. The ball released from rest and the ball initially moving downward.

The function for the height difference between the ball released from rest (ball 1) and the ball initially moving downward (ball 3) is the difference between their two equations of motion:

$$y_{f,1} - y_{f,3} = -\frac{1}{2}gt^2 - \left(-ut - \frac{1}{2}gt^2\right) = -\frac{1}{2}gt^2 + ut + \frac{1}{2}gt^2 = ut. \tag{6}$$

D. Find the formula for the maximum height above the ground reached by the ball initially moving upward.

The maximum height of ball 2, y_{max}, occurs when its velocity reaches zero. Therefore, we first need to find the time at which the velocity is zero. Using the equation $v_{f,y} = v_{0,y} - gt$, we can find this time. We set $v_{f,y}$ to zero, and substitute our known initial velocity:

$$0 = u - gt. \tag{7}$$

Adding gt to both sides and then dividing both sides by g, we see that the time at which the ball reaches its max height t_{max} is then:

$$t_{max} = \frac{u}{g}. \tag{8}$$

Here, we can then substitute this time into our original equation of motion for ball 2 to find its max height:

$$y_{2,max} = y_2(t = t_{max}) = u\left(\frac{u}{g}\right) - \frac{1}{2}g\left(\frac{u}{g}\right)^2 = \frac{u^2}{g} - \frac{gu^2}{2g^2} = \frac{u^2}{g} - \frac{1}{2}\frac{u^2}{g} = \frac{1}{2}\frac{u^2}{g}. \tag{9}$$

If we wanted to skip this whole process, we could also use the "shortcut" equation, $v^2 - v_0^2 = 2a\Delta y$, with $\Delta y = y_f - y_0 = y_f = y_{max}$, and $v^2 = 0$:

$$0 - u^2 = 2(-g)(y_m ax), \tag{10}$$

$$\Rightarrow y_{max} = \frac{u^2}{2g} \tag{11}$$

E. Substitute the height found in part d into the height equations for

1. the ball released from rest and solve for t to find when these balls reach that height. What do you find? What do these results mean

In other words, when does the ball released from rest (ball 1) reach the max height of ball 2? Substituting the equation for the max height of ball 2 into the equation of motion for ball 1,

$$y_{f,1} = -\frac{1}{2}gt_{max}^2, \tag{12}$$

$$\frac{u^2}{2g} = -\frac{1}{2}gt_{max}^2. \tag{13}$$

Solving for t_{max},

$$-\frac{2u^2}{2g} = gt_{max}^2, \tag{14}$$

$$\Rightarrow -\frac{u^2}{g^2} = t_{max}^2. \tag{15}$$

Taking the square root of both sides:

$$t_{max} = \pm\sqrt{-\frac{u^2}{g^2}}. \tag{16}$$

We have a problem. The expression inside of the square root is negative. This means that the time we are solving for is not a real number. The math is telling us here that the ball released from rest will *never* reach the max height of ball 2.

2. the ball initially moving downward and solve for t to find when these balls reach that height. What do you find? What do these results mean?

Repeating the same process for ball 3, we have the original equation of motion:

$$y_{f,3} = -ut_{max} - \frac{1}{2}gt_{max}^2. \tag{17}$$

Setting $y_{f,3}$ equal to the equation for the max height of ball 2,

$$\frac{u^2}{2g} = -ut_{max} - \frac{1}{2}gt_{max}^2. \tag{18}$$

Solving for t_{max}, we can first add both terms on the right hand side to both sides (to get rid of the negatives):

$$\frac{1}{2}gt_{max}^2 + ut_{max} + \frac{u^2}{2g}. \tag{19}$$

To get this in a more familiar form, we can then multiply both sides by 2 and divide by g to get:

$$0 = t_{max}^2 + 2\frac{u}{g}t_{max} + \left(\frac{u}{g}\right)^2. \tag{20}$$

From here, we can see that this is a polynomial (specifically a quadratic), that we are relatively familiar with solving. We *could* use the quadratic equation for this, but we can also make an observation about the form it is currently in. Remember that for any polynomial in the following form:

$$(x + a)^2, \tag{21}$$

We can foil it out in the standard way to get:

$$(x + a)(x + a) = x^2 + 2ax + a^2. \tag{22}$$

We can see that our equation (Eq. 42) looks very similar to the form on the ride hand side if we take $a = \frac{u}{g}$. Therefore, it must be true that:

$$\Rightarrow 0 = \left(t_{max} + \frac{u}{g}\right)^2, \tag{23}$$

Here we can see that we have *something* plus *something*. The best way to make that addition zero is to make them equal and opposite. Therefore,

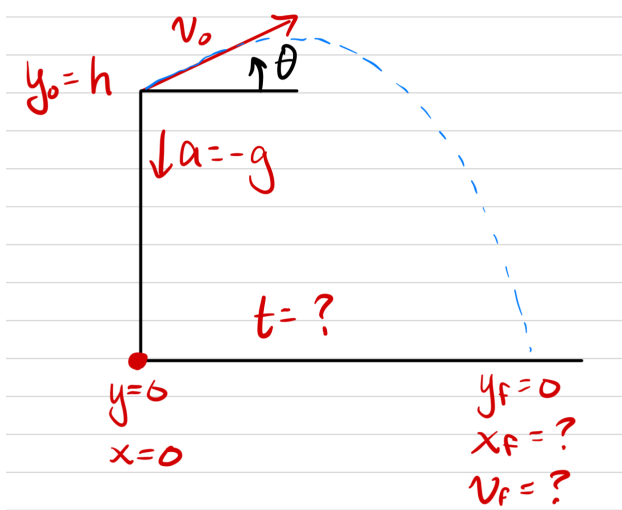
$$t_{max} = -\frac{u}{g}. \tag{24}$$

Just like before, this is a problem. We have a negative time, which is not physically significant. The math is once again telling us that (in this specific situation), ball 3 will *never* reach the max height of ball 2.

II. A projectile is launched from height h at speed v_0 and angle θ above the horizontal and lands on the ground.

A. Draw a diagram of the initial situation. Show axis directions and the location of the origin.

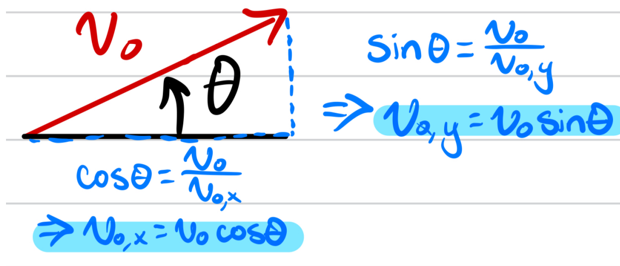
Here we know the change in y position ($y_0 = h$ and $y_f = 0$ because it lands on the ground), we can set our initial x position, x_0, to be zero. We are given the angle of launch, θ. We also know the initial velocity v_0, and the acceleration in the y direction ($a_y = -g$). A diagram for this case would then be:



B. Find the formula for the maximum height reached by the projectile.

At the maximum height of the projectile, its velocity in the y direction will go to zero. Therefore, our first step to find the maximum height reached by the ball should be to find the time, t_{max}, at which the ball reaches it max height (in other words, find the time that the y velocity becomes zero.

*A note on the initial velocity. We are given the initial velocity v_0. This velocity is a combination of both the velocity in the x direction ($v_{0,x}$) *and* the velocity in the y direction ($v_{0,y}$). If we want to analyze the motion of the ball in just one of these directions (much, much easier than both directions at once), then we will need to break v_0 into its *components*. If we know the initial velocity as well as the angle at which it was thrown, we can find the components using our right triangle trigonometry. The following diagram shows how we find $v_{0,y} = v_0 \sin\theta$ and $v_{0,x} = v_0 \cos\theta$ using these methods:



Using the given equations on the equation sheet, we know that in general (for the y direction):

$$v_{f,y} = v_{0,y} - gt. \tag{25}$$

We want to find t_{max}, where $v_{f,y} = 0$. Therefore,

$$0 = v_{0,y} - gt_{max}. \tag{26}$$

To solve for t_{max} our first step is to add gt_{max} to both sides to get:

$$gt_{max} = v_{0,y}. \tag{27}$$

Then we divide both sides by g and get a final expression for t_{max}:

$$t_{max} = \frac{v_{0,y}}{g}. \tag{28}$$

From here, we can substitute this time expression into the equation for the y position as a function of time. This equation is:

$$y_f = y_0 + v_{0,y}t - \frac{1}{2}gt^2. \tag{29}$$

Substituting $t = t_{max}$:

$$y_{max} = y_0 + v_{0,y}\left(\frac{v_{0,y}}{g}\right) - \frac{1}{2}g\left(\frac{v_{0,y}}{g}\right)^2. \tag{30}$$

This can simplify to:

$$y_{max} = y_0 + \frac{v_{0,y}^2}{g} - \frac{1}{2}\frac{v_{0,y}^2}{g} = y_0 + \frac{v_{0,y}^2}{g} - \frac{1}{2}\frac{v_{0,y}^2}{g}. \tag{31}$$

The final two terms are similar to taking $x - 1/2x$, which is equal to $1/2x$. Therefore, this simplifies to:

$$y_{max} = y_0 + \frac{1}{2}\frac{v_{0,y}^2}{g}. \tag{32}$$

We know that $y_0 = h$. We also know that $v_{0,y} = v_0 \sin\theta$. Therefore, our final expression for the max height of the ball is then:

$$y_{max} = y_0 + \frac{1}{2}\frac{(v_0 \sin\theta)^2}{g}. \tag{33}$$

We could have also found the maximum height straight away using this "shortcut" formula that was given on the previous problem set's equation sheet:

$$v_{f,y}^2 - v_{0,y}^2 = 2a(y_f - y_0). \tag{34}$$

As with before, we want to solve for y_f, where $v_{f,y} = 0$. Putting in our zero:

$$-v_{0,y}^2 = 2a(y_{max} - y_0). \tag{35}$$

Our first step to get y_{max} on its own is to divide both sides by $2a$:

$$\frac{-v_{0,y}^2}{2a} = y_{max} - y_0. \tag{36}$$

Then we can add y_0 to both sides to get:

$$y_{max} = -\frac{v_{0,y}^2}{2a} + y_0. \tag{37}$$

Substituting known "values" ($v_{0,y} = v_0 \sin\theta$, $a = -g$, and $y_0 = h$), we get:

$$y_{max} = -\frac{(v_0 \sin\theta)^2}{2(-g)} + h. \tag{38}$$

This simplifies to be the exact same equation as before,

$$y_{max} = h + \frac{(v_0 \sin\theta)^2}{2g}. \tag{39}$$

Notice that the math is telling us the max height is higher than its original position, h. This should make logical sense. When you throw a ball at any angle above the horizon, its max height should always be higher than when it started.

C. Find the formula for the time at which the projectile reaches the top of its arc.

If you chose the first method for the last part, you have already found this equation to be:

$$t_{max} = \frac{v_{0,y}}{g} = \frac{v_0 \sin\theta}{g}. \tag{40}$$

If you chose to use the "shortcut" equation in the last step, finding t_{max} is still possible, just a little more involved. Using that method, we already know an equation for y_{max}. To find the t_{max}, we need to find the time at which $y_f = y_{max}$. As before, the general equation for the position, y of a projectile as a function of time is:

$$y_f = y_0 + v_{0,y}t + \frac{1}{2}at^2 \tag{41}$$

From here, we need to substitute $y_f = y_{max}$, $y_0 = h$, $v_{0,y} = v_0 \sin\theta$, and $a = -g$, and solve for t_{max}:

$$h + \frac{(v_0 \sin\theta)^2}{2g} = h + v_0 \sin\theta t_{max} - \frac{1}{2}gt_{max}^2. \tag{42}$$

Right away we can see that there is an h on both sides. We can subtract h from both sides to be left with:

$$\frac{(v_0 \sin\theta)^2}{2g} = v_0 \sin\theta t_{max} - \frac{1}{2}gt_{max}^2. \tag{43}$$

For simplicity's sake, we can also multiply both sides by 2 and subtract the term on the left side on both sides. We then get:

$$2v_0 \sin\theta t_{max} - gt_{max}^2 - \frac{(v_0 \sin\theta)^2}{g} = 0. \tag{44}$$

We can also divide both sides by $-g$ and do a bit of rearranging to get this in a familiar form:

$$t_{max}^2 - 2\left(\frac{v_0 \sin\theta}{g}\right)t_{max} + \left(\frac{v_0 \sin\theta}{g}\right)^2 = 0 \tag{45}$$

From here, we can see that this is another polynomial. Like before, we can "defoil" this if we take $a = -\frac{v_0 \sin\theta}{g}$. It then must be true that:

$$t_{max}^2 - 2\left(\frac{v_0 \sin\theta}{g}\right)t_{max} + \left(\frac{v_0 \sin\theta}{g}\right)^2 = \left(t_{max} - \frac{v_0 \sin\theta}{g}\right)^2. \tag{46}$$

Therefore, what we are *really* trying to solve is:

$$\left(t_{max} - \frac{v_0 \sin\theta}{g}\right)^2 = 0. \tag{47}$$

As it turns out, this equation (which should have *two* solutions), actually has only one unique solution (if you don't believe me, test Eq. 44 with the quadratic formula!). Without doing any intense math, we can use logic to find what that solution is. If we want the left hand side of the equation to be zero, then what we really want is for the terms inside of the parenthesis to be zero. Therfore, we are trying to solve:

$$t_{max} - \frac{v_0 \sin\theta}{g} = 0. \tag{48}$$

Our first (and only step) is then to add $\frac{v_0 \sin\theta}{g}$ to both sides. We then get a familiar result:

$$t_{max} = \frac{v_0 \sin\theta}{g}. \tag{49}$$

As expected, this method gets us the same result as the first method, just with a little bit more dirty work involved.

D. Find the formula for the projectiles velocity $\vec{v} = (v_x, v_y)$ at the top of its arc.

At the top of the arc, we know that the velocity in the y direction is zero ($v_y = 0$). We also know that there is no acceleration in the x direction. This means that there is nothing changing the velocity in the x direction. Therefore, $v_x = v_{0,x}$ at any point. The velocity, $\vec{v}$ is then:

$$\vec{v} = (v_{0,x}, 0). \tag{50}$$

E. Find the formula for the vertical component of the projectiles velocity just as it reaches the ground.

Just like with the max, we can do this with two different methods. We can either find the time it takes to reach the ground and then substitute into a velocity equation, or we can use the shortcut velocity equation. Lets do the first method first.

For this case, we know the change in y position ($y_0 = h$ and $y_f = 0$), we know the initial y velocity ($v_{0,y} = v_0 \sin\theta$), and we know the acceleration ($a = -g$). Therefore, we want an equation that can utilize these terms with t as our unknown. The following equation can achieve this:

$$y_f = y_0 + v_{0,y}t + \frac{1}{2}at^2. \tag{51}$$

We know that $y_f = 0$ so,

$$0 = y_0 + v_{0,y}t + \frac{1}{2}at^2. \tag{52}$$

Before we substitute anything else, we should solve for t first. To make this more familiar, we can multiply by 2 and divide by a:

$$t^2 + \frac{2v_{0,y}}{a}t + \frac{2y_0}{a} = 0. \tag{53}$$

Just like the previous question, this is a quadratic equation. Unlike the previous question, we can't "unfoil" this equation as easily. Our best bet is to use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{54}$$

In this case, $a = 1$, $b = \frac{2v_{0,y}}{a}$, and $c = \frac{2y_0}{a}$. Therefore,

$$t = \frac{-\frac{2v_{0,y}}{a} \pm \sqrt{\left(\frac{2v_{0,y}}{a}\right)^2 - 4(1)\left(\frac{2y_0}{a}\right)}}{2(1)} = \frac{-\frac{2v_{0,y}}{a} \pm \sqrt{4\left(\frac{v_{0,y}}{a}\right)^2 - 4\left(\frac{2y_0}{a}\right)}}{2}. \tag{55}$$

Here, we see that we can pull out a 4 from both terms under the square root:

$$t = \frac{-\frac{2v_{0,y}}{a} \pm \sqrt{4\left[\left(\frac{v_{0,y}}{a}\right)^2 - \left(\frac{2y_0}{a}\right)\right]}}{2}. \tag{56}$$

That 4 can then come out of the square root as a 2 ($\sqrt{4} = 2$):

$$t = \frac{-\frac{2v_{0,y}}{a} \pm 2\sqrt{\left(\frac{v_{0,y}}{a}\right)^2 - \left(\frac{2y_0}{a}\right)}}{2}. \tag{57}$$

Finally, the twos in the numerators and the denominator can cancel out to give us:

$$t = -\frac{v_{0,y}}{a} \pm \sqrt{\left(\frac{v_{0,y}}{-g}\right)^2 - \left(\frac{2y_0}{a}\right)}. \tag{58}$$

From here we can substitute our known "values":

$$t = -\frac{v_0 \sin\theta}{-g} \pm \sqrt{\left(\frac{v_0 \sin\theta}{a}\right)^2 - \frac{2h}{-g}} = \frac{v_0 \sin\theta}{g} \pm \sqrt{\left(\frac{v_0 \sin\theta}{-g}\right)^2 + \frac{2h}{g}}. \tag{59}$$

As much as we would like to further simplify this equation, this is as simplified as this can get. One final step is to figure out which part of the $\pm$ we want to keep. Logically, we can see that we want to keep the + and get rid of the minus, because the + is the only operation that would give us a positive time. We are then left with our final time equation:

$$t = \frac{v_0 \sin\theta}{g} + \sqrt{\left(\frac{v_0 \sin\theta}{-g}\right)^2 + \frac{2h}{g}}. \tag{60}$$

Now that we know the time, we can substitute it back into an equation that will tell us the velocity at this time. The best equation for this is:

$$v_{f,y} = v_{0,y} + at. \tag{61}$$

We don't have to do any algebra to isolate our variable, so we can go ahead and substitute our known values:

$$v_{f,y} = v_0 \sin\theta - g\left(\frac{v_0 \sin\theta}{g} + \sqrt{\left(\frac{v_0 \sin\theta}{-g}\right)^2 + \frac{2h}{g}}\right). \tag{62}$$

If we distribute the $-g$:

$$v_{f,y} = v_0 \sin\theta - \frac{g v_0 \sin\theta}{g} - g\sqrt{\left(\frac{v_0 \sin\theta}{-g}\right)^2 + \frac{2h}{g}}. \tag{63}$$

Remembering that $g = \sqrt{g^2}$, we can put that g into the square root:

$$v_{f,y} = v_0 \sin\theta - v_0 \sin\theta - \sqrt{\frac{g^2 v_0^2 \sin^2\theta}{g^2} + \frac{2g^2 h}{g}}. \tag{64}$$

A few more simplification steps give:

$$v_{f,y} = -\sqrt{(v_0 \sin\theta)^2 + 2gh}. \tag{65}$$

This is the final equation for the velocity of the projectile as it hits the ground.

We could have also used the "shortcut" equation:

$$v_{f,y}^2 - v_{0,y}^2 = 2a(y_f - y_0). \tag{66}$$

Adding $v_{0,y}^2$ to both sides and taking the square root:

$$v_{f,y} = \pm\sqrt{2a(y_f - y_0) + v_{0,y}^2}. \tag{67}$$

Substituting knowns,

$$y_{f,y} = \pm\sqrt{2(-g)(0-h) + (v_0 \sin\theta)^2}. \tag{68}$$

Or, after simplifying:

$$v_{f,y} = \pm\sqrt{(v_0 \sin\theta)^2 + 2gh}. \tag{69}$$

As before, we need to decide with part of the $\pm$ to keep. In this case, we know that the final y velocity should be negative since the projectile is moving down at the time it reaches the ground. Therefore, just like with the other method:

$$v_{f,y} = -\sqrt{(v_0 \sin\theta)^2 + 2gh}. \tag{70}$$

F. Find the formula for the time at which the projectile lands on the ground.

Just like before, if we went the "long" route, we have already found the time to be:

$$t = \frac{v_0 \sin\theta}{g} + \sqrt{\left(\frac{v_0 \sin\theta}{-g}\right)^2 + \frac{2h}{g}}. \tag{71}$$

If we chose to go the shortcut route, we can still get this result. Using the shortcut, we found that:

$$v_{f,y} = -\sqrt{(v_0 \sin\theta)^2 + 2gh}. \tag{72}$$

We can plug this into the following equation to find the time:

$$v_{f,y} = v_{0,y} + at. \tag{73}$$

To solve for time, we can subtract $v_{0,y}$ from both sides and divide by a to get:

$$t = \frac{v_{f,y} - v_{0,y}}{a}. \tag{74}$$

Substituting our knowns:

$$t = \frac{-\sqrt{(v_0 \sin\theta)^2 + 2gh} - v_0 \sin\theta}{-g}. \tag{75}$$

All of our minus signs cancel out. We can also distribute the g in the denominator into the square root ($g = \sqrt{g^2}$):

$$t = \frac{v_0 \sin\theta}{g} + \sqrt{\left(\frac{v_0 \sin\theta}{g}\right)^2 + \frac{2gh}{g^2}}. \tag{76}$$

This finally simplifies to give us the expected equation for the time:

$$t = \frac{v_0 \sin\theta}{g} + \sqrt{\left(\frac{v_0 \sin\theta}{g}\right)^2 + \frac{2h}{g}}. \tag{77}$$

G. Find the formula for the horizontal distance the projectile travels to the top of its arc.

Recall from a previous part that the time to reach the max is:

$$t_{max} = \frac{v_0 \sin\theta}{g}. \tag{78}$$

We can substitute this into an equation that tells us the x position as a function of time:

$$x_f = x_0 + v_{0,x} t_{max} + \frac{1}{2} a t_{max}^2. \tag{79}$$

Since $a = 0$ in the x direction, this is just:

$$x_f = x_0 + v_{0,x} t_{max}. \tag{80}$$

We can also see that we set $x_0 = 0$ in our picture, so we really just have:

$$x_f = v_{0,x} t_{max}. \tag{81}$$

Substituting t_{max} and $v_{0,x} = v_0 \cos\theta$:

$$x_f = \frac{v_0 \cos\theta \cdot v_0 \sin\theta}{g} = \frac{v_0^2 \sin\theta \cos\theta}{g}. \tag{82}$$

H. Find the formula for the horizontal distance the project travels to land on the ground.

Just like before, we already know that the time it takes for the ball to reach the ground is:

$$t = \frac{v_0 \sin\theta}{g} + \sqrt{\left(\frac{v_0 \sin\theta}{g}\right)^2 + \frac{2h}{g}}. \tag{83}$$

We can substitute this into the equation we used before to find the final distance:

$$x_f = v_{0,x} t. \tag{84}$$

Substituting gives:

$$x_f = (v_0 \cos\theta) \left(\frac{v_0 \sin\theta}{g} + \sqrt{\left(\frac{v_0 \sin\theta}{g}\right)^2 + \frac{2h}{g}} \right). \tag{85}$$

Simplifying this can get pretty messy pretty fast. This is an acceptable form for a final answer, but if we wanted to simplify here's what we could do:

$$x_f = \frac{v_0^2 \sin\theta \cos\theta}{g} + v_0 \cos\theta \sqrt{\left(\frac{v_0 \sin\theta}{g}\right)^2 + \frac{2h}{g}}. \tag{86}$$

Pulling the $v_0 \cos\theta$ into the square root:

$$x_f = \frac{v_0^2 \sin\theta\cos\theta}{g} + \sqrt{\frac{v_0^2\cos^2\theta v_0^2\sin^2\theta}{g^2} + \frac{2v_0^2\cos^2\theta h}{g}}. \tag{87}$$

We can multiply the second term in the square root by $\frac{g}{g} = 1$ to get:

$$x_f = \frac{v_0^2 \sin\theta\cos\theta}{g} + \sqrt{\frac{v_0^2\cos^2\theta v_0^2\sin^2\theta}{g^2} + \frac{2v_0^2\cos^2\theta gh}{g^2}} \tag{88}$$

$$\Rightarrow x_f = \frac{v_0^2 \sin\theta\cos\theta}{g} + \sqrt{\frac{v_0^4\cos^2\theta\sin^2\theta}{g^2} + \frac{2v_0^2\cos^2\theta gh}{g^2}}. \tag{89}$$

We can also multiply the second term in the square root by $\frac{v_0^2}{v_0^2} = 1$ to get:

$$x_f = \frac{v_0^2 \sin\theta\cos\theta}{g} + \sqrt{\frac{v_0^4\cos^2\theta\sin^2\theta}{g^2} + \frac{2v_0^4\cos^2\theta gh}{v_0^2 g^2}}. \tag{90}$$

This seems rather daunting, but this allows us to pull out a $\frac{v_0^4\cos^2\theta}{g^2}$ term in the square root:

$$x_f = \frac{v_0^2 \sin\theta\cos\theta}{g} + \sqrt{\frac{v_0^4\cos^2\theta}{g^2}\left(\sin^2\theta + \frac{2gh}{v_0^2}\right)} \tag{91}$$

We can then take the square root of the term we pulled out to get:

$$x_f = \frac{v_0^2 \sin\theta\cos\theta}{g} + \frac{v_0^2\cos\theta}{g}\sqrt{\sin^2\theta + \frac{2gh}{v_0^2}}. \tag{92}$$

Our final simplification step is to pull out a $\frac{v_0^2\cos\theta}{g}$ term to get a final equation for x_f:

$$x_f = \frac{v_0^2\cos\theta}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2gh}{v_0^2}}\right). \tag{93}$$

This is a possible simplified form of this equation.

I. *The projectile is a car tire fired from a hobbyists catapult at a speed of 20.0 m/s at an angle of 50 degrees above horizontal from a height of 2.50 m above the ground. How far away horizontally from the launch does it land on the ground?*

We have numbers! Luckily for us, we did all the heavy lifting before, and we have an equation for the horizontal distance of the car tire with only known values. This problem tells us the following:

$$v_0 = 20.0 \text{ m/s},$$

$$\theta = 50^\circ,$$

$$h = 2.50 \text{ m},$$

And of course, $g = 9.8$ m/s^2. Since we already have an equation, we can now just plug all these numbers into the equation we found before:

$$x_f = \frac{(20.0)^2 \cos(50^\circ)}{9.8} \left(\sin(50^\circ) + \sqrt{\sin^2(50^\circ) + \frac{2(9.8)(2.50)}{(20.0)^2}} \right). \tag{94}$$

Putting all this into a calculator gives us a final horizontal distance of $x_f = 42.2$ meters.