# Discussion 10: Angular Momentum and Oscillations

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***In the interest of getting these solutions out before the exam, I opted to not draw as many pictures this time, except for when it is necessary. I still encourage you to draw pictures (or look at/label the ones provided)***

**1.** A *ballistic pendulum* is a venerable device used to indirectly measure cannon muzzle velocities. The cannonball is fired horizontally and is immediately caught and held by a pivoted catcher assembly. The catcher swings upward, allowing the initial cannonball velocity to be deduced. The ball of mass $m$ is fired at speed $v$ and caught at the center of mass of a pivoted catcher of mass $M$, hanging a distance $R$ below its frictionless pivot. The catcher has a moment of inertia $I$ about the pivot. The catcher swings to a maximum angle of $\theta$ from the vertical, raising its center of mass by $h$.

**1a)** Use conservation of angular momentum to find the angular speed $\omega$ of the assembly immediately after catching the cannonball (treat the cannonball as a point particle).

Conservation of angular momentum tells us that:

$$L_1 = L_2, \tag{1}$$

Where $L_1$ is the initial angular momentum and $L_2$ is the final angular momentum. We have two different expressions for angular momentum, depending on what we have. Our first one relates the angular momentum to the linear momentum ($p = mv$) and the distance from the axis of rotation, $R$:

$$\vec{L} = \vec{p}R = m\vec{v}R. \tag{2}$$

Our second equation relates moment of inertia of our rotating body, $I$, to its angular velocity, $\omega$:

$$L = I\vec{\omega} \tag{3}$$

Our initial angular momentum is made up of only the objects that have angular motion. Our pendulum is at rest at the beginning, so the only angular momentum involved in the system is the angular momentum of the cannonball. We are told that it is has some mass $m$ and is launched with some velocity $v$. We also know that the cannonball is some distance $R$ from the pivot. We have all the information we need to be able to say that the angular momentum of the cannonball is:

$$L_{ball} = mvR. \tag{4}$$

We can then say that our initial angular momentum is:

$$L_1 = L_{ball} = mvR. \tag{5}$$

Our final angular momentum (once the pendulum catches the cannonball) is the angular momentum of both the cannonball and the pendulum. Since we are rotating around an axis after catching, we will have some angular velocity, $\omega$. We are also given that the moment of inertia of the pendulum is $I_{pend} = I$. Luckily, moments of inertia simply added together to find the moment of inertia for a system, $I_{sys}$. In our case then, $I_{sys} = I_{ball} + I_{pend} = I_{ball} + I$. We can then use our second definition of angular momentum to find the final angular momentum:

$$L_2 = I_{sys}\omega = (I_{ball} + I)\omega. \tag{6}$$

Now we need to find an equation for the moment of inertia for the ball around the rotation axis. The key thing to keep in mind here is that the rotation axis is the point where the pendulum is attached. All that we have in the system is the pendulum and the ball rotating around the axis. There is nothing inside of circle that the system is tracing out. In other words, we have a circular path with no material inside the circle–a hoop. We then need to use the moment of inertia equation for a hoop:

$$I_{hoop} = mR^2. \tag{7}$$

This is not the moment of inertia for the *whole* system, just the ball (the pendulum itself has other material that contributes to its moment of inertia). Therefore,

$$I_{ball} = mR^2, \tag{8}$$

$$I_{sys} = I_{ball} + I_{pend} = mR^2 + I. \tag{9}$$

Therefore, our final angular momentum is:

$$L_2 = (mR^2 + I)\omega. \tag{10}$$

Our full angular momentum conservation equation then reads:

$$mvR = (mR^2 + I)\omega. \tag{11}$$

Our goal was to find $\omega$, so let's solve for that. If we divide both sides by $mR^2 + I$:

$$\omega = \frac{mvR}{mR^2 + I} \tag{12}$$

**1b)** Use conservation of mechanical energy to find the maximum height $h$ attained by the catcher.

Conservation of mechanical energy says:

$$E_1 = E_2. \tag{13}$$

We know that $E$ is the sum of kinetic energy and any sources of potential energy. Since there are no springs involved, we can say that the only potential involved is gravitational. Therefore,

$$K_1 + U_{g1} = K_2 + U_{g2}. \tag{14}$$

This time, instead of state 1 being the launch of the ball, state 1 is going to be the point where the ball is caught by the pendulum. This way, we can set state 2 as the point where the pendulum reaches its maximum height. Our only kinetic energy after catching the ball is rotational ($K_{rot} = \frac{1}{2}I\omega^2$). Therefore,

$$\frac{1}{2}I_{sys}\omega_1^2 + mgh_1 = \frac{1}{2}I_{sys}\omega_2^2 + mgh_2. \tag{15}$$

Our mass term, $m$, should *actually* be $m + M$. This is because the mass that we are referring to in these equations is the combined mass of the pendulum and the ball:

$$\frac{1}{2}I_{sys}\omega_1^2 + (m+M)gh_1 = \frac{1}{2}I_{sys}\omega_2^2 + (m+M)gh_2. \tag{16}$$

There are a few other simplification steps we can take. If we set our zero height to be at the bottom of the pendulum/ball path, then our initial gravitational potential energy is then zero (in other words, $h_1 = 0$). Similarly, we can set our final kinetic energy to be zero as well. This is because we are looking for the *maximum* height of the system. Just like in our 2-dimensional kinematics situation, our maximum height is going to be when we are next at rest in our path (in other words, $\omega_2 = 0$). Therefore,

$$\frac{1}{2}I_{sys}\omega_1^2 = (m+M)gh_2. \tag{17}$$

If we divide both sides of the equation by $(m+M)g$, we see that the maximum height is:

$$h_2 = \frac{I_{sys}\omega_1}{2g(m+M)}. \tag{18}$$

We found before that the angular velocity of our system was $\omega = \frac{mvR}{mR^2 + I}$. Therefore,

$$\omega^2 = \frac{(mvR)^2}{(mR^2+I)^2} \tag{19}$$

Putting this into our equation for $h_2$:

$$h_2 = \frac{I_{sys}}{2g(m+M)}\frac{(mvR)^2}{(mR^2+I)^2} \tag{20}$$

Finally, our moment of inertia for the system, $I_{sys}$ was determined to be $I_{sys} = mR^2 + I$. Therefore,

$$h_2 = \frac{mR^2 + I}{2g(m+M)} \frac{(mvR)^2}{(mR^2 + I)^2}. \tag{21}$$

We can see that we have:

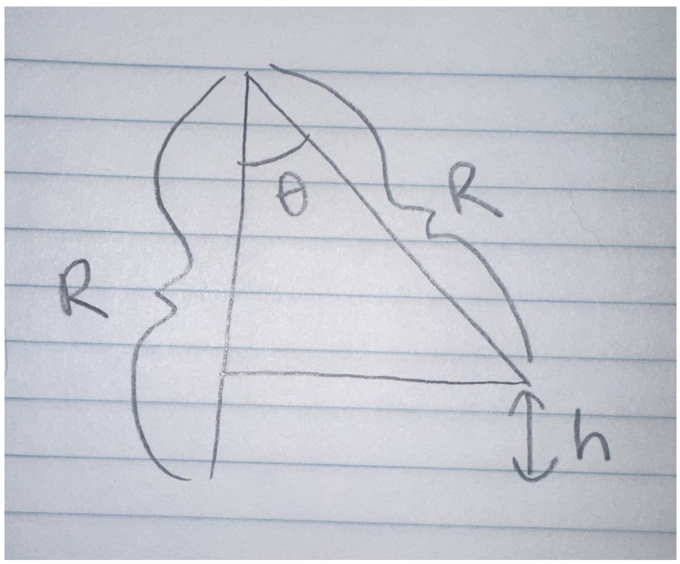
$$\frac{mR^2 + I}{(mR^2 + I)^2} = \frac{1}{mR^2 + I}. \tag{22}$$

Therefore, our final equation for the maximum height is:

$$h_2 = \frac{(mvR)^2}{2g(m+M)(mR^2 + I)} \tag{23}$$

**1c)** What is the maximum angle $\theta$ attained by the catcher?

This problem requires a new picture:




We can see a few different triangles here. If we want to find the maximum angle of the pendulum, we want to relate the maximum height to the angle. We can see that the left leg of our triangle is $R - h$. We also know that the hypotenuse of our right triangle is *also* $R$ (it doesn't look like it in the diagram, but think about it; the rod that the pendulum is attached to isn't stretching or compressing. So at the maximum height, the pendulum is still the same distance $R$ from the axis). Therefore, we can say that the left leg is also equal to $R\cos\theta$. Therefore,

$$R - h = R\cos\theta. \tag{24}$$

If we divide both sides by $R$,

$$\frac{R}{R} - \frac{h}{R} = \cos\theta, \tag{25}$$

$$\Rightarrow \cos\theta = 1 - \frac{h}{R}. \tag{26}$$

If we take the inverse cosine, we find that the angle is:

$$\theta = \cos^{-1}\left(1 - \frac{h}{R}\right). \tag{27}$$

**1d)** Let's try it with some numbers. The cannonball's mass is 5.00 kg, its initial speed is 200.0 m/s, the mass of the catcher is 200.0 kg, the catcher is 5.00 m from the pivot, and the moment of inertia of the catcher is 5020 kg m$^2$.

Now we can put in numbers! We are given the cannonball mass is $m = 5.00$ kg, and its initial speed is $v = 200.0$ m/s. We are also given that the pendulum catcher has a mass $M = 200.0$ kg, and is $R = 5.00$ meters from the axis of rotation. Finally, we are given that the moment of inertia of the catcher is $I = 5020$ kg m$^2$.

**1.d.i.** What is the angular speed of the assembly after the catch?

Looking at our angular speed equation:

$$\omega = \frac{mvR}{mR^2 + I}. \tag{28}$$

These are all knowns. We can plug in numbers:

$$\omega = \frac{(5.00)(200.0)(5.00)}{(5.00)(5.00)^2 + (5020)} = 0.972 \text{ rad/s}. \tag{29}$$

**1.d.ii.** How high above its starting height does the catcher swing?

We can use our height equation:

$$h_2 = \frac{(mvR)^2}{2g(m+M)(mR^2+I)}. \tag{30}$$

Putting in numbers:

$$h_2 = \frac{[(5.00)(200.0)(5.00)]^2}{2(9.8)(5.00+200.0)[(5.00)(5.00)^2 + (5020)]} = 1.209 \text{ m}. \tag{31}$$

**1.d.iii.** What is the maximum angle $\theta$ attained by the catcher?

We can find this value using our angle equation found before:

$$\theta = \cos^{-1}\left(1 - \frac{h}{R}\right) \tag{32}$$

We just found $h = 1.209$. Therefore,

$$\theta = \cos^{-1}\left(1 - \frac{1.209}{5.00}\right) = 40.7°. \tag{33}$$

**2.** A 42 kg snowman slides along frictionless ice at a speed of 2.10 m/s toward a coil spring with a spring constant of 1700 N/m, as 1. The snowman runs into the spring, latching onto it, as 2.

**2a)** Knowing that mechanical energy is conserved, what is the maximum *compression* of the spring?

Conservation of mechanical energy tells us:

$$E_1 = E_2. \tag{34}$$

Since our height isn't changing, we don't have to consider gravitational potential. However, we *are* working with a spring in this problem, so we need to consider spring potential:

$$U_s = \frac{1}{2}kx^2, \tag{35}$$

Where $k$ is the spring constant (given to be 1700 N/m in our problem) and $x$ is how far the spring has been compressed. Our conservation of energy equation is then:

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2. \tag{36}$$

At the beginning (just before the snowman hits the spring) the spring is not compressed, so $x_1 = 0$. When the spring has reached its maximum compression, we know that the snowman has lost all of its kinetic energy to compress the spring. Therefore, $v_2 = 0$. Keeping these things in mind:

$$\frac{1}{2}mv_1^2 = \frac{1}{2}kx_2^2. \tag{37}$$

We can now solve for $x_2$ to find the maximum compression of the spring. We can divide both sides by 1/2 (or multiply both sides by 2):

$$mv_1^2 = kx_2^2. \tag{38}$$

Dividing both sides by $k$:

$$x_2^2 = \frac{mv_1^2}{k} \tag{39}$$

Finally, taking the square root gives:

$$x_2 = \sqrt{\frac{mv_1^2}{k}}. \tag{40}$$

Now we can put in numbers. We are given that the mass of the snowman is $m = 42$ kg, and that the initial velocity is $v = 2.10$ m/s. We are also given that the spring constant is $k = 1700$ N/m. Therefore,

$$x_2 = \sqrt{\frac{(42)(2.10)^2}{1700}} = 0.33 \text{ m}. \tag{41}$$

**2b)** What is the *amplitude* of the snowman's oscillation?

The *amplitude*, $A$, of this oscillation is the maximum compression, which we just found to be $A = x_2 =$ 0.33 m.

**2c)** What is the *period* of the snowman's oscillation?

The period of this oscillation can be found with the following equation:

$$T = \frac{2\pi}{\omega}. \tag{42}$$

The only problem here is that we don't know $\omega$. Luckily, we can find it with the information we know. Our equations tell us that:

$$\omega = \sqrt{\frac{k}{m}}. \tag{43}$$

We know both of these values, so we can do some substitution to find $\omega$:

$$\omega = \sqrt{\frac{(1700)}{(42)}} = 6.36 \text{ rad/s}. \tag{44}$$

Now we can put this into the period equation:

$$T = \frac{2\pi \text{ [rad]}}{(6.36 \text{ [rad/s]})} = 0.988 \text{ s}. \tag{45}$$

**2d)** What is the maximum *speed* of the snowman in its oscillation?

In this situation, our maximum speed should be the initial speed. As the snowman keeps going after this point, the spring starts to slow it down. That means our maximum speed is $v_{max} = 2.10$ m/s.

**2e)** What is the maximum *acceleration* of the snowman in its oscillation?

The acceleration can be given by:

$$a = -A\omega^2 \cos(\omega t). \tag{46}$$

The $\cos(\omega t)$ term can have values ranging from -1 to 1. If we want to find the *maximum* acceleration, we want to set our $\cos(\omega t)$ term equal to 1:

$$\cos(\omega t) = 1. \tag{47}$$

This means that we want the terms inside of the cosine to go to zero ($\cos(0) = 1$). The only way this can happen is if $t = 0$. This tells us that the snowman experiences the most acceleration at $t = 0$, which is *right when it hits the spring.* If $\cos(\omega t) = 1$, then:

$$a = -A\omega^2. \tag{48}$$

Putting our $A$ and $\omega$ terms into the equation:

$$a = -(0.33)(6.36)^2 = -13.36 \text{ m/s}^2. \tag{49}$$

In terms of magnitudes, this means that the snowman is *slowing down* by 13.36 meters per second every second.

**2f)** What is the maximum *net force* acting on the snowman in its oscillation?

Here we can use our good friend Newton's Second Law:

$$\sum F = ma. \tag{50}$$

We know both $m$ and $a$, so our maximum net force, $\sum F$ is then:

$$\sum F = (42)(-13.36) = -561.1 \text{ N}, \tag{51}$$

Which means that the highest amount of net force that the snowman experiences is 561.1 N, applied opposite to its motion.