Discussion 10: Static torque, moment of inertia, and angular momentum

Summary

Rotational work and energy

Kinetic energy

The rotational kinetic energy of a rotating body is $K_r = ½\, I\omega^2$.

The motion of any rigid body can be described as a combination of translation of its center of mass and rotation about its center of mass. Its total kinetic energy is
$K = K_t + K_r = ½\, \mathrm{m}v^2 + ½\, I\omega^2$.

Work

As a force $\vec{F}$ acting along a displacement $\vec{s}$ does work $\vec{F} \cdot \vec{s}$, a torque $\vec{\tau}$ acting along an angular displacement $\vec{\theta}$ does work $\vec{\tau} \cdot \vec{\theta}$. This allows an interpretation of the unit for torque, N·m, as equivalent to a J/rad.

Angular momentum

Angular momentum of a point mass with respect to a reference point is $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{r}$ is the vector from the reference point to the point mass, and $\vec{p}$ is the point mass's momentum.

The angular momentum of a rigid rotor is $I\vec{\omega}$, with I being the body's moment of inertia for rotation about the reference point.

The torque applied to a body is the rate of change of its angular momentum, $\vec{\tau} = \Delta\vec{L}/\Delta t$.

Conservation of angular momentum

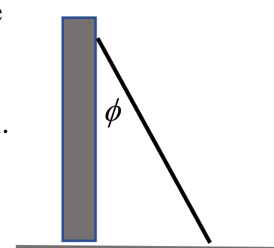
If a body receives no external torques, its angular momentum does no change.

If its moment of inertia changes, its angular velocity changes inversely. $I_1\vec{\omega}_1 = I_2\vec{\omega}_2$

Problems

1. A ladder rests against a wall at angle ϕ from vertical. The mass of the ladder is m and its length is L. The ladder's coefficient of static friction with the ground is μ_s, while the wall it leans against is frictionless. We are concerned about the ladder sliding down the wall.

   There are several ways to consider this problem. We will evaluate all torques about the point where the bottom of the ladder contacts the ground. Other reference points can be used as well; another intuitive point is the ladder's center of mass.



   A. There are four forces acting on the ladder.

      1. Its weight $\vec{w} = m\vec{g}$
      2. Normal force from the ground $\vec{N}_1$

3. Normal force from the wall $\vec{N}_2$
4. Friction against the ground $f = \mu_s N_1$

i. Draw a force diagram for the ladder showing these forces.

ii. When the ladder is static, these forces add to zero in both the horizontal and vertical directions. Forces $\vec{w}$ and $\vec{N}_1$ are purely vertical, while $\vec{f}$ and $\vec{N}_2$ are purely horizontal. Thus, we know that $N_1 = w = mg$. At the limit of static friction, $f = \mu_s mg$. A static situation requires $\vec{N}_2 = f$.

iii. Write out the *x*- and *y*-components of these four forces.

B. Now we need to find the torques resulting from these forces.

i. The torques resulting from forces $\vec{N}_1$ and $\vec{f}$ are both zero. Why? ________________

ii. Find the magnitude of the torque produced by the weight of the ladder. Is this torque positive (counterclockwise) or negative (clockwise)?

iii. Find the magnitude of the torque produced by the normal force $\vec{N}_2$. Is this torque positive (counterclockwise) or negative (clockwise)?

C. For the ladder to rest at angle ϕ, the net torque on it must be zero. Use this fact to find the formula for the coefficient of friction μ_s in terms of the angle ϕ.

2. In the past, before child safety was as high a priority as it is now, many playgrounds contained carousels. (You can watch a video of the carousel that used to stand in Laramie's Harbon Park at https://youtu.be/BFBGr0GsAPY?si=CKSsZivkSRTRvfx2.) These were circular horizontal platforms that would rotate about their centers. They were usually equipped with rails to make them easy to push, and to help riders hold on when the carousels spun rapidly.

A. Suppose a playground carousel has a radius of 1.50 meters and a mass of 200. kilograms. We can approximate it as a uniform cylinder to estimate its moment of inertia. (The formula for the moment of inertia of a uniform cylinder rotating about its principal axis is $\frac{1}{2} MR^2$.) What is the approximate moment of inertia of the carousel?

B. Carousels weren't built so that we could watch them spin—they were made to ride. Suppose four 30.-kilogram children ride on the carousel. They are evenly-spaced around its rim. (OK, not quite at the rim, because then they'd fall off. Let's say they stand 0.15 meters from the rim, at 1.35 meters from the axis.) Approximate the children as point masses. What amount do the children add to the moment of inertia of the carousel?

There's no point in just standing on a carousel—ride it while it spins. Suppose that the carousel with the four riders rotates to make one complete circle every 3.5 seconds.

C. What is the angular momentum of the rotating carousel (including riders)?

D. What is the kinetic energy of the rotating carousel (including riders)?

E. What is the centripetal acceleration of the riders?

Suppose that the riders move together toward the center of the carousel, to a distance of 0.90 meters from the axis.

F. Is the angular momentum of the carousel (including riders) conserved?

G. What is the new period of rotation of the carousel?

H. What is the new kinetic energy of the rotating carousel (including riders)?

I. What is the new centripetal acceleration of the riders?