PHYS 1210 Discussion 10. Physical Pendulum

Solutions

Physical pendulum with square bob of sides a

The pendulum has mass M and sides a. The axis of rotation is distance L from the center of mass. We want to find the distance L giving the highest oscillation frequency.

First, I'll go over finding the frequency of a physical pendulum.

A physical pendulum is a type of torsional oscillator, which has a torque given by the angular Hoolke's law $\tau = -\kappa\theta$. Here, κ is the torque constant of the torsional spring, how hard it tries to turn back when twisted; and θ is how far (radians) it is twisted from rest. The differential equation for such a system is $I\alpha = -\kappa\theta$; by its form we can see that its general solution is of the same form as the general solution of a simple harmonic oscillator, with the angular frequency ω given by $\omega^2 = \kappa/I$.

To apply the torsional oscillator results to a physical pendulum, we need to find the appropriate κ and I.

To find κ, we find the torque on the pendulum. The torque is the tangential component of the net force on the bob, multiplied by the radius L. The only force acting on the pendulum that has a nonzero tangential component is its weight. The tangential component of weight is $-Mg\sin\theta$, so the torque is $-MgL\sin\theta$. This is *almost* in the $-\kappa\theta$ form that we need, except that it has $\sin\theta$ instead of θ. Fortunately, for small θ, $\sin\theta \approx \theta$, so we can approximate $\tau = -MgL\theta$, giving us $\kappa = MgL$.

Next, we need to find I, the moment of inertia of the physical pendulum. For this we use the parallel axis theorem, $I = I_{cm} + ML^2$. The moment of inertia of a square of mass M and sides of length a rotated about its principal axis (which passes through its center of mass) is $1/6\ Ma^2$. Thus $I = 1/6\ Ma^2 + ML^2$.

Finally, we can get an expression for the angular frequency of the pendulum.

$$\omega^2 = \kappa/I = \frac{MgL}{1/6\ Ma^2 + ML^2} = \frac{gL}{a^2/6 + L^2}$$

We are asked to find the highest frequency $f = \omega/2\pi$. We don't really need to write the full formula for f, because the conditions giving the highest ω will also give the highest f. For that matter, we don't even need to write the full formula for ω, because the conditions giving the highest ω^2 will also give the highest ω (and highest f).

To find the L giving the highest ω^2 is a min-max problem. That will be the L where $d(\omega^2)/dL = 0$. (Strictly, we should verify that ω^2 is concave downward there. I'll just assume that.) We take the derivative of ω^2 with respect to L:

$$\frac{d}{dL}(\omega^2) = g\frac{(a^2/6 + L^2) - L(2L)}{(a^2/6 + L^2)^2} = g\frac{a^2/6 + L^2 - 2L^2}{(a^2/6 + L^2)^2} = g\frac{a^2/6 - L^2}{(a^2/6 + L^2)^2}$$

We find the L at which this derivative equals zero.

$$\begin{aligned} 0 &= g\frac{a^2/6 - L^2}{(a^2/6 + L^2)^2} \\ 0 &= a^2/6 - L^2 \\ L^2 &= a^2/6 \\ L &= a/\sqrt{6} \end{aligned}$$

We need to be sure that the denominator $\left(a^2/6 + L^2\right)^2$ isn't zero, to avoid singularities. We see that can't happen if a and L are both real and at least one of them is nonzero. The square bob is not a point mass, so

$a > 0$.

It's worth noting that $\sqrt{6}$ is greater than 2, so the pivot of this pendulum is *inside* the square bob.