PHYS 1210 Discussion 14

Brief Solutions

Binary Star System

We start with some round numbers. The mass of one star is $m_1 = 4 \times 10^{30}$ kilograms, about 2 solar masses; the other star is $m_2 = 2 \times 10^{30}$ kilograms, about one solar mass; and the constant distance between the stars is $D = 3 \times 10^{12}$ meters, about 20 astronomical units.

1. Distance from Star 1 to the center of mass r_1

   First we need to find the center of mass. The most straightforward way to do this is to set Star 1 at $x_1 = 0$ and Star 2 at $x_2 = D$. Then the center of mass will be located at $x_{cm} = r_1$.

   $$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{0\ m_1 + D\ m_2}{m_1 + m_2} = D \frac{m_2}{m_1 + m_2}$$

   So $r_1 = D\ m_2/(m_1 + m_2)$.

2. Distance from Star 2 to the center of mass r_2

   Using the same reasoning as for r_1, we could solve this by setting Star 2 at $x_2 = 0$ and Star 1 at $x_1 = D$; we'll obtain $r_2 = x_{cm} = D\ m_1/(m_1 + m_2)$. We could also find it by starting with $r_2 = D - r_1 = D - D\ m_2/(m_1 + m_2)$ and simplifying.

   One way that I think is slick is to set the position of the center of mass at $x_{cm} = 0$, then to solve the system of simultaneous equations

   $$0 = \frac{-m_1 r_1 + m_2 r_2}{m_1 + m_2} \qquad \text{and} \qquad r_1 + r_2 = D.$$

   All of these give the same answer, and the two formulas interconvert if you exchange the indices 1 and 2.

3. Tangential orbital speed of Star 1

   The physics to answer this question is that the centripetal force of the circular orbit of radius r_1 is the gravitational attraction between stars 1 and 2 at distance D.

   $$\begin{aligned} \frac{m_1 v_1^2}{r_1} &= \frac{G m_1 m_2}{D^2} \\ v_1^2 &= G m_2 \frac{r_1}{D^2} \\ &= G m_2 \frac{D\ m_2}{m_1 + m_2} \Big/ D^2 \\ &= \frac{G}{D} \frac{m_2^2}{m_1 + m_2} \end{aligned}$$

   If we want v_1, we can just take the square root. But that's not pretty.

4. Tangential orbital speed of Star 2

   The same reasoning as above again reveals that exchanging the indices gives the correct formula.

   $$v_2^2 = \frac{G}{D} \frac{m_1^2}{m_1 + m_2}$$

5. Total kinetic energy

   The kinetic energy of each star is $1/2\ mv^2$, giving us more justification for stopping our algebra at the stage of v^2.

$$
\begin{aligned}
K &= 1/2\ m_1 v_1^2 + 1/2\ m_2 v_2^2 \\
&= 1/2\ m_1 \frac{G}{D} \frac{m_2^2}{m_1+m_2} + 1/2\ m_2 \frac{G}{D} \frac{m_1^2}{m_1+m_2} \\
&= \frac{G}{2D} \frac{m_1 m_2}{m_1+m_2}(m_2+m_1) \\
&= \frac{G\ m_1 m_2}{2D}
\end{aligned}
$$

   It's a good check at this point to verify that the units of this expression actually are energy units, because this certainly doesn't look like $1/2\ mv^2$. Checking the form of the equation reveals that it looks like a gravitational potential energy formula, except for that factor of 2 in the denominator. Which leads us to...

6. Gravitational potential energy

$$
U_g = -\frac{Gm_1 m_2}{D^2}
$$

   The orbital kinetic energy is half the magnitude of the gravitational potential energy. As we saw in the Keplerian case before, "orbit is halfway to the stars." Even though, in this case, these orbiting bodies *are* stars.

7. Orbital angular momentum

   We haven't found angular velocity yet (that's a little bit later), so we'll want to use the angular momentum formula $\vec{r} \times \vec{p}$.

$$
\begin{aligned}
L &= L_1 + L_2 \\
&= r_1 m_1 v_1 + r_2 m_2 v_2 \\
&= D\frac{m_2}{m_1+m_2} m_1 \sqrt{\frac{G}{D}\frac{m_2^2}{m_1+m_2}} + D\frac{m_1}{m_1+m_2} m_2 \sqrt{\frac{G}{D}\frac{m_1^2}{m_1+m_2}} \\
&= \frac{m_2^2 m_1}{m_1+m_2}\sqrt{\frac{D^2 G/D}{m_1+m_2}} + \frac{m_1^2 m_2}{m_1+m_2}\sqrt{\frac{D^2 G/D}{m_1+m_2}} \\
&= \frac{m_1 m_2}{m_1+m_2}\sqrt{\frac{GD}{m_1+m_2}}(m_2+m_1) \\
&= m_1 m_2 \sqrt{\frac{GD}{m_1+m_2}}
\end{aligned}
$$

   That's not exactly pretty, but it's not too ugly, either. It's a worthwhile exercise to verify that the units are angular momentum units.

8. Angular velocity ω

   It's convenient to use $\omega = v/r$; we should be able to use either v_1/r_1 or v_2/r_2 for the same answer. Because the formulas for v_1 and v_2 have square roots in them, let's make it cleaner by looking for ω^2.

$$
\begin{aligned}
\omega^2 &= v_1^2/r_1^2 \\
&= \frac{G}{D}\frac{m_2^2}{m_1+m_2}\bigg/\left(\frac{D\ m_2}{m_1+m_2}\right)^2 \\
&= \frac{G}{D^3}\frac{m_2^2}{m_2^2}\frac{(m_1+m_2)^2}{m_1+m_2} \\
&= \frac{G\ (m_1+m_2)}{D^3}
\end{aligned}
$$

If we really want to know ω, we can just take the square root.

9. Orbital period

   Again, I'll find the square. It's neater.

   We can get T from ω by $T = 2\pi/\omega$, so $T^2 = 4\pi^2/\omega^2$.

$$
T^2 = \frac{4\pi^2\ D^3}{G(m_1+m_2)}
$$

   which preserves the Keplerian law that the square of the period T is proportional to the cube of the separation D.

10. Orbital period for $m_1 = 4 \times 10^{30}$ kg, $m_2 = 2 \times 10^{30}$ kg, $D = 3 \times 10^{12}$ m

$$
\begin{aligned}
T^2 &= \frac{4\pi^2\ (3 \times 10^{12}\ \mathrm{m})^3}{(6.674 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(6 \times 10^{30}\ \mathrm{kg})} = 2.6635 \times 10^18\ \mathrm{s}^2 \\
T &= 1.6320 \times 10^9\ \mathrm{s}
\end{aligned}
$$

   That number doesn't tell us much, so we can convert it to longer time scales. We get 4.5334×10^5 hours, which is 1.8889×10^4 days, which is 51.7 years. For what it's worth, the orbital period of the Sirius binary star system is 50.1 years.