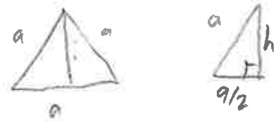
Equilateral Triangle

Altitude of an equilateral Δ with sides of length a

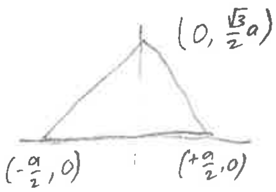


$h^2 + (a/2)^2 = a^2$

$h^2 = a^2 - a^2/4 = a^2(1 - \frac{1}{4}) = a^2(\frac{3}{4})$

$\boxed{h = (\frac{\sqrt{3}}{2})a}$

Center of mass of three equal point masses at the vertices of an equilateral Δ



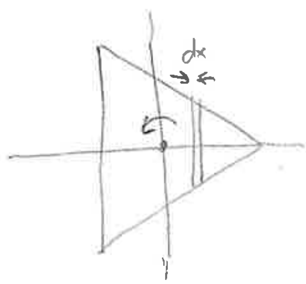
$$x_{cm} = \frac{-\frac{a}{2}m + 0m + \frac{a}{2}m}{3m} = \frac{0}{3m} = 0$$

$$y_{cm} = \frac{0m + \frac{\sqrt{3}}{2}am + 0m}{3m} = \boxed{\frac{\sqrt{3}}{6}a} \text{ from base. } \boxed{\frac{\sqrt{3}}{3}a \text{ from vertex}}$$

Compared to the altitude, this is $\frac{1}{3}(\frac{\sqrt{3}}{2}a) = \frac{1}{3}h$

Area of the equilateral Δ $\frac{1}{2}a\frac{\sqrt{3}}{2}a = \boxed{\frac{\sqrt{3}}{4}a^2 = A}$ so its density is $\boxed{\sigma = \frac{M}{A} = \frac{4M}{\sqrt{3}a^2}}$

Moment of inertia about an axis perpendicular to the plane through center of mass



x from $-\frac{\sqrt{3}}{6}a$ to $+\frac{\sqrt{3}}{3}a$

$$y = \frac{a}{3} - \frac{x}{\sqrt{3}}$$

verify when $x = -\frac{\sqrt{3}}{6}a$, $y = \frac{a}{3} - \frac{-a}{6} = a/2$

when $x = \frac{+\sqrt{3}}{3}$, $y = \frac{a}{3} - \frac{a}{3} = 0$ ✓

mass of the differential element $dm = \sigma\, 2y\, dx$
$= 2\sigma\left(\frac{a}{3} - \frac{x}{\sqrt{3}}\right)dx$

area density $\sigma = \frac{M}{A} = \frac{M}{\sqrt{3}/4\, a^2} = \frac{4M}{\sqrt{3}a^2}$

Differential element of moment of inertia: $dI = dI_{cm} + x^2 dm$

$$dI_{cm} = \frac{1}{12}(2y)^2 dm = \frac{1}{3}y^2 dm = \frac{1}{3}\left(\frac{a}{3} - \frac{x}{\sqrt{3}}\right)^2 dm$$

$$dI = \left[\frac{1}{3}\left(\frac{a}{3} - \frac{x}{\sqrt{3}}\right)^2 + x^2\right]dm = \left[\frac{1}{3}\left(\frac{a}{3} - \frac{x}{\sqrt{3}}\right)^2 + x^2\right]\sigma\, 2y\,dx = 2\sigma\left[\frac{1}{3}\left(\frac{a}{3} - \frac{x}{\sqrt{3}}\right)^2 + x^2\right]\left(\frac{a}{3} - \frac{x}{\sqrt{3}}\right)dx$$

$$= 2\sigma\left[\frac{1}{3}\left(\frac{a}{3} - \frac{x}{\sqrt{3}}\right)^3 + \frac{a}{3}x^2 - \frac{x^3}{\sqrt{3}}\right]dx$$

$$I = \frac{2}{3}\sigma\int\left(\frac{a}{3} - \frac{x}{\sqrt{3}}\right)^3 dx + \frac{2\sigma a}{3}\int x^2 dx - \frac{2\sigma}{\sqrt{3}}\int x^3 dx$$

The second and third integrals are easy. The first takes a simple substitution

$$u = \underbrace{\frac{a}{3} - \frac{x}{\sqrt{3}}}_{\text{this is } y}, \quad dx = -\sqrt{3}\,du$$

$\Rightarrow$ Perhaps leave this as y, and integrate from $y = \frac{a}{2}$ to $y = 0$

$$I = \frac{-2\sqrt{3}\sigma}{3}\,\frac{u^4}{4} + \frac{2\sigma a}{3}\,\frac{x^3}{3} - \frac{2\sigma}{\sqrt{3}}\,\frac{x^4}{4}\Bigg|_{x=-\frac{\sqrt{3}}{6}a}^{x=\frac{\sqrt{3}}{3}a}$$

$$= \frac{-\sqrt{3}\sigma}{6}\left(\frac{a}{3}-\frac{x}{\sqrt{3}}\right)^4 + \frac{2\sigma a}{9}x^3 - \frac{\sigma}{2\sqrt{3}}x^4\Bigg|_{x=\frac{-\sqrt{3}}{6}a}^{x=\frac{\sqrt{3}}{3}a}$$

$$= \frac{-\sqrt{3}\sigma}{6}\left[\left(\frac{a}{3}-\frac{a}{3}\right)^4 - \left(\frac{a}{3}+\frac{a}{6}\right)^4\right] + \frac{2\sigma a}{9}\left(\frac{3\sqrt{3}a^3}{27}+\frac{3\sqrt{3}a^3}{216}\right) - \frac{\sigma}{2\sqrt{3}}\left(\frac{9}{81}a^4 - \frac{9}{1296}a^4\right)$$

$$= \frac{-\sqrt{3}\sigma}{6}\left[-\left(\frac{a}{2}\right)^4\right] + \frac{2\sigma}{9}a^4\sqrt{3}\left(\frac{8}{72}+\frac{1}{72}\right) - \frac{\sigma a^4}{2\sqrt{3}}\left(\frac{1}{9}-\frac{1}{144}\right)$$

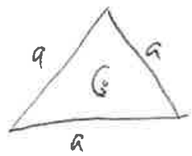
$$= \frac{\sqrt{3}\sigma a^4}{96} + \frac{2\sqrt{3}\sigma a^4}{9}\,\frac{1}{8} - \frac{\sigma a^4}{2\sqrt{3}}\left(\frac{16-1}{144}\right)$$

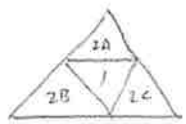
$$= a^4\sigma\left(\frac{\sqrt{3}}{96}+\frac{\sqrt{3}}{36}-\frac{15\sqrt{3}}{864}\right) = a^4\sigma\sqrt{3}\left(\frac{1}{96}+\frac{1}{36}-\frac{15}{864}\right)$$

$$= a^4\,\frac{4M}{\sqrt{3}a^2}\,\sqrt{3}\left(\frac{9}{864}+\frac{24}{864}-\frac{15}{864}\right)$$

$$= Ma^2\,4\left(\frac{18}{864}\right) = \boxed{\frac{1}{12}Ma^2}$$

Moment of Inertia of an Equilateral Triangle by Geometry





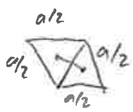
divide into four congruent equilateral triangles

1, 2A, 2B, 2C $\qquad I = I_1 + 3I_2$

Triangle 1 has mass $\frac{1}{4}M$ side $a/2$

Assuming I is of the form cMa^2 Then $I_1 = c\frac{M}{4}\left(\frac{a}{2}\right)^2 = I/16$

Triangles 2 have $I_2 = I_1 + \frac{M}{4}d^2$ where d is distance to their cm



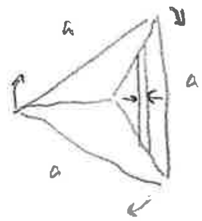
The distance from one edge to centroid is $\frac{1}{3}\frac{\sqrt{3}}{2}\frac{a}{2} = \frac{\sqrt{3}}{12}a$

The distance d is twice that, or $\frac{\sqrt{3}}{6}a$

So $I_2 = I_1 + \frac{M}{4}\left(\frac{\sqrt{3}}{6}a\right)^2 = \frac{I}{16} + \frac{M}{4}\frac{3}{36}a^2 = \frac{I}{16} + \frac{M}{4}\frac{1}{12}a^2 = \frac{I}{16} + \frac{1}{48}Ma^2$

$$I = I_1 + 3I_2$$
$$= \frac{I}{16} + 3\left(\frac{I}{16} + \frac{1}{48}Ma^2\right)$$
$$= \frac{I}{16} + \frac{3I}{16} + \frac{1}{16}Ma^2$$
$$I = \frac{I}{4} + \frac{1}{16}Ma^2$$
$$\left(I - \frac{I}{4}\right) = \frac{1}{16}Ma^2$$
$$\frac{3}{4}I = \frac{1}{16}Ma^2$$
$$I = \frac{4}{3}\frac{1}{16}Ma^2 = \frac{1}{12}Ma^2$$

simple triangular Thirds of an Equilateral Triangle



This will find a Third

$dI_{cm} = \frac{1}{12}\, dm\, y^2$ limits x from 0 to $\frac{\sqrt{3}}{6}a$ y from 0 to $a/2$

$y = \frac{a/2}{\sqrt{3}a/6}x = \frac{a}{\sqrt{3}a}\frac{6}{2}x = \frac{3}{\sqrt{3}}x = \sqrt{3}x$ $\frac{dy}{dx} = \sqrt{3}$ so $dx = \frac{dy}{\sqrt{3}}$

$dm = \sigma dA = \sigma(2y)dx = 2\sigma y\, dx = \frac{2}{\sqrt{3}}\sigma y\, dy$

$= \sigma(2\sqrt{3})x\, dx = 2\sqrt{3}\sigma x\, dx$

$A = \frac{1}{2}a\frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{4}a^2$

$\sigma = \frac{4M}{\sqrt{3}a^2}$

$dI = dI_{cm} + x^2 dm = \frac{1}{12}(2y)^2 dm + x^2 dm = \left(\frac{1}{3}y^2 + x^2\right)dm$

$= \left(\frac{1}{3}3x^2 + x^2\right)dm = (x^2 + x^2)\,dm = (2x^2)\,2\sqrt{3}\sigma x\, dx$

$= 4\sqrt{3}\sigma x^3 dx$

$= \left(\frac{1}{3}y^2 + \frac{y^2}{3}\right)dm = \left(\frac{2}{3}y^2\right)\frac{2}{\sqrt{3}}\sigma y\, dy$

$= \frac{4}{3\sqrt{3}}\sigma y^3 dy$

$I = \int dI = 4\sqrt{3}\sigma\frac{x^4}{4}\Big|_{x=0}^{x=\frac{\sqrt{3}}{6}a}$

$= \sqrt{3}\sigma x^4\Big|_{x=0}^{x=\sqrt{3}a/6} = \sqrt{3}\sigma\frac{1}{144}a^4$

now $\sigma = \frac{4M}{\sqrt{3}a^2}$

$= \frac{\sqrt{3}}{144}\frac{4M}{\sqrt{3}a^2}a^4$

$= \frac{1}{36}Ma^2$

Triple This is $\frac{1}{12}Ma^2$ ✓

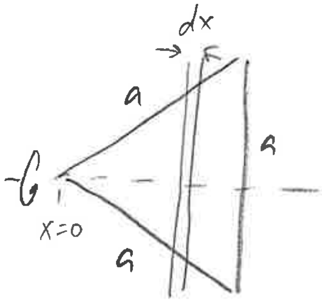
$I = \int dI = \frac{4}{3\sqrt{3}}\sigma\frac{y^4}{4}\Big|_{y=0}^{y=a/2}$

$= \frac{\sigma}{3\sqrt{3}}y^4\Big|_{y=0}^{y=a/2} = \frac{\sigma}{3\sqrt{3}}\frac{a^4}{16}$

$= \frac{1}{48\sqrt{3}}\frac{4M}{\sqrt{3}a^2}a^4$

$= \frac{1}{12\cdot 3}Ma^2 = \frac{1}{36}Ma^2$

Moment of Inertia of an Equilateral Triangle Rotated about a midline



variable of integration: x from 0 to $\frac{\sqrt{3}}{2}a$

$dI = \frac{1}{12} dm\,(2y)^2 \qquad y = \frac{x}{\sqrt{3}}$

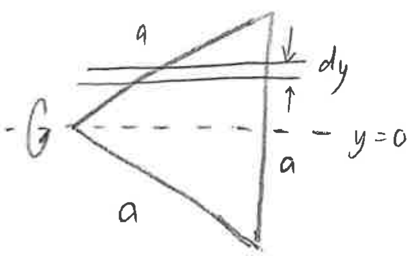
$dm = \sigma\, 2y\, dx$

so $dI = \frac{1}{12}\sigma (2y)^3 dx = \frac{1}{12}\sigma\left(\frac{2}{\sqrt{3}}x\right)^3 dx$

$= \frac{1}{12}\sigma \frac{8}{3\sqrt{3}} x^3 dx = \frac{2\sigma}{9\sqrt{3}} x^3 dx$

$$I = \int dI = \frac{2\sigma}{9\sqrt{3}} \frac{x^4}{4}\Bigg|_{x=0}^{x=\frac{\sqrt{3}}{2}a} = \frac{\sigma}{18\sqrt{3}} \frac{9}{16}a^4 = \frac{1}{32\sqrt{3}}\sigma a^4$$

$$= \frac{1}{32\sqrt{3}} \frac{4M}{\sqrt{3}a^2} a^4 = \frac{1}{24} M a^2$$



variable of integration: y from 0 to $a/2$

This will get half the triangle; double the result

Mass of the element $= \left(\frac{\sqrt{3}}{2}a - x\right) dy\, \sigma = dm$

$x = \sqrt{3}\, y$

$dm = \left(\frac{\sqrt{3}}{2}a - \sqrt{3}y\right)\sigma\, dy$

Then $dI = y^2 dm = \sigma y^2 \left(\frac{\sqrt{3}}{2}a - \sqrt{3}y\right) dy = \sqrt{3}\sigma y^2 \left(\frac{a}{2} - y\right) dy$

$= \sqrt{3}\sigma \left(\frac{a}{2}y^2 - y^3\right) dy$

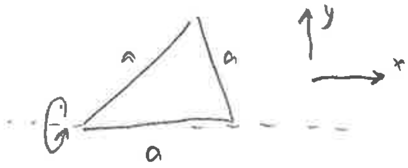
$$I = \sqrt{3}\sigma\left[\frac{a}{2}\int y^2 dy - \int y^3 dy\right] = \sqrt{3}\sigma\left[\frac{a}{6}y^3 - \frac{1}{4}y^4\right]\Bigg|_{y=0}^{y=a/2}$$

$$= \sqrt{3}\sigma\left[\frac{a}{6}\frac{a^3}{8} - \frac{1}{4}\frac{a^4}{16}\right] = \sqrt{3}\sigma\left(\frac{1}{48} - \frac{1}{64}\right)a^4 = \sqrt{3}\sigma\left(\frac{4}{192} - \frac{3}{192}\right)a^4$$

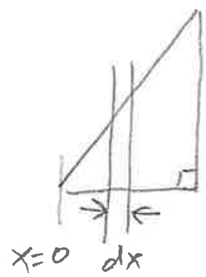
$$= \sqrt{3}\sigma\left(\frac{1}{192}\right)a^4 = \frac{\sqrt{3}}{192}\frac{4M}{\sqrt{3}a^2}a^4 = \frac{1}{48}Ma^2$$

Double this to get $\boxed{I = \frac{1}{24}Ma^2}$

Equilateral Triangle Rotated About an Edge



Integrating over x: find I of right-triangle half, double it



$dm = \sigma y\, dx \qquad \sigma = \frac{4M}{\sqrt{3}\, a^2}$

$y = \sqrt{3}\, x$

$dI = \frac{1}{3} y^2 dm = \frac{1}{3} y^2 \sigma y\, dx = \frac{\sigma}{3} y^3 dx = \frac{\sigma}{3} (\sqrt{3}\, x)^3 dx$

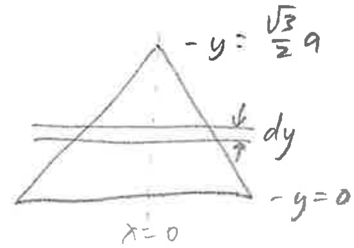
$= \frac{\sigma}{3} 3\sqrt{3}\, x^3 dx = \sqrt{3}\, \sigma x^3 dx$

$I = \int dI = \sqrt{3}\sigma \int x^3 dx = \sqrt{3}\, \sigma \left.\frac{x^4}{4}\right|_0^{a/2} = \sqrt{3}\, \sigma \frac{a^4}{64}$

$= \sqrt{3} \frac{4M}{\sqrt{3}\, a^2} \frac{a^4}{64} = \frac{1}{16} Ma^2$

Double this for the whole equilateral triangle $\boxed{I = \frac{1}{8} Ma^2}$

Integrating over y:



length = $2x$

$x = \frac{a}{2} - \frac{y}{\sqrt{3}}$

$dA = 2x\, dy$

$dm = 2\sigma x\, dy = 2\sigma\left(\frac{a}{2} - \frac{y}{\sqrt{3}}\right) dy$

$dI = y^2 dm = y^2 2\sigma \left(\frac{a}{2} - \frac{y}{\sqrt{3}}\right) dy$

$= 2\sigma \left(\frac{a}{2} y^2 - \frac{y^3}{\sqrt{3}}\right) dy$

$I = \int_{y=0}^{y=\frac{\sqrt{3}}{2}a} dI = 2\sigma\left[\frac{a}{2}\int y^2 dy - \frac{1}{\sqrt{3}}\int y^3 dy\right] = 2\sigma\left[\frac{a}{6} y^3 - \frac{1}{4\sqrt{3}} y^4\right]_{y=0}^{y=\frac{\sqrt{3}}{2}a}$

$= 2\sigma\left[\frac{a}{6} \frac{3\sqrt{3}}{8} a^3 - \frac{1}{4\sqrt{3}} \frac{9}{16} a^4\right] = 2\sigma\left[\frac{\sqrt{3}}{16} a^4 - \frac{3\sqrt{3}}{64} a^4\right]$

$= 2\sigma \sqrt{3}\, a^4 \left(\frac{4}{64} - \frac{3}{64}\right) = 2\sigma \sqrt{3}\, a^4 \frac{1}{64} = \frac{\sqrt{3}}{32} \sigma a^4$

$= \frac{\sqrt{3}}{32} \frac{4M}{\sqrt{3}\, a^2} a^4 = \boxed{\frac{1}{8} Ma^2}$ same answer both ways