PHYS 1210 Exam 3

Brief Solutions

1. Grandpa's grinding wheel

A. Initial angular speed

The initial angular speed is 72.0 rev/min. Our task here is to convert to radians per second. The conversion factors we need to use are 60 s = 1 min and 2π radians = 1 revolution.

$$\omega_0 = 72.0\ \frac{\text{rev}}{\text{min}} \cdot \frac{\text{min}}{60\ \text{s}} \cdot \frac{2\pi\ \text{rad}}{\text{rev}} = \frac{12\pi}{5}\frac{\text{rad}}{s} = 2.4\pi\frac{\text{rad}}{s} = 7.540\ \frac{\text{rad}}{s}$$

B. Angular acceleration

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{0-\omega_0}{\Delta t} = \frac{2.4\pi\ \text{rad/s}}{15.0\ \text{s}} = 0.16\pi\ \frac{\text{rad}}{\text{s}^2} = 0.5027\ \text{rad/s}^2$$

C. Revolutions completed

The best kinematic equation to use is $\Delta\theta = 1/2\ (\omega + \omega_0)\ \Delta t$. We can use that using the angular velocities we found in radians per second, or we can just do the whole thing in revolutions and minutes to avoid converting to radians and back.

$$\Delta\theta = \frac{1}{2}(\omega+\omega_0)\Delta t = \frac{1}{2}(72\ \text{rev/min})(0.25\ \text{min}) = 9\ \text{rev}$$

D. Average angular speed

$$\omega = \Delta\theta/\Delta t = 9\ \text{rev}/0.25\ \text{min} = 36\ \text{rev/min} = 3.770\ \text{rad/s}$$

E. Moment of inertia

Treat the grinding wheel as a uniform cylinder. The formula for the moment of inertia of that shape about its principal axis is $1/2\ MR^2$. With $M = 25.0$ kg and = 0.20 m, that gives us $1/2\ \text{kg}\cdot\text{m}^2$.

F. Net torque

We know of two ways to think about torque: as a cross product of force and a lever arm, or as an influence causing an angular acceleration. The latter is what works here. $\tau = I\alpha = (1/2\ \text{kg}\cdot\text{m}^2)(0.16\pi\ \text{rad/s}^2) = 0.08\pi\ \text{N}\cdot\text{m} = 0.251\ \text{N}\cdot\text{m}$.

G. Torque from the axe

For the last question, I used the second sense of torque. For this question, we use the first. The friction force is tangential, giving a torque with a magnitude of just $fr = (3.5\ \text{N})(0.20\ \text{m}) = 0.7\ \text{N}\cdot\text{m}$.

H. Work to maintain rotation

This question could also have been about the work done on the grinding wheel by the friction from the axe. The work done by the axe was negative, so Grandpa had to do positive work to maintain the wheel's speed and kinetic energy. Work done by a torque is the torque multiplied by the angular displacement $\Delta\theta = 36$ rev $= 72\pi$ rad; $W = \tau\Delta\theta = 158.35$ J.

2. Rolling spare tire

A. Translational kinetic energy at 5.5 m/s

$K = 1/2mv^2 = 1/2(13.8 \text{ kg})(5.5 \text{ m/s})^2 = 208.7$ J.

B. Rotational kinetic energy at 5.5 m/s

Rotational kinetic energy is $1/2\ I\omega^2$. We are given $I = 0.370 \text{ kg} \cdot \text{m}^2$, but we need to find ω. The tire rolls without slipping, so $\omega = v/r = 26.19$ rad/s. Then $K_{rot} = 126.90$ J.

3. Leaning against a wall

A. Torque exerted by her weight

This is a clockwise (negative) torque. The radius vector is from her feet to her center of mass 1.10 meters away. The force is straight down with magnitude 500 newtons. (The 500 newtons is her *weight* mg, not her mass.) The angle from vector $\vec{r}$ to vector $\vec{F}$ is $-150°$.Then

$$\tau = \vec{r} \times \vec{F} = rF\sin(-150°) = (1.10 \text{ m})(500 \text{ N})(-0.5) = -275 \text{ N} \cdot \text{m}$$

B. Lever arm for her weight

The lever arm of her weight about her feet is the horizontal separation between her center of gravity and her feet, or $(1.10 \text{ m})\cos(60°) = 0.55$ m.

C. Torque from force F_N

We can't calculate the torque from the force and lever arm, because we don't know the force. Instead, we use our knowledge that the leaning woman is in mechanical equilibrium, so the wall must provide a torque exactly cancelling the torque from her weight, $= 275 \text{ N} \cdot \text{m}$.

D. Lever arm for the force from the wall

For this force, the line of action is horizontal, so its lever arm is the *vertical* height of her shoulders, $1.50 \text{ m} \sin 60° = 1.299$ m.

E. Magnitude of F_N

The easy way to find this is to use the fact that the torque is the force times the lever arm, $\tau = FL$, so $F = \tau/L = (275 \text{ N} \cdot \text{m})/(1.299 \text{ m}) = 211.7$ N.

F. Straightening up

If she becomes more vertical, she will push less against the wall. In reaction, the wall pushes less against her.

4. Hunting with boleadoras

A. Moment of inertia about the initial center of mass

One ball of mass m is a distance $4/3\ D$ from the center. Two balls of mass m are $2/3\ D$ from the center. Just add their moments together:

$$I_1 = m(4/3\ D)^2 + 2\left[m(2/3\ D)^2\right] = \frac{16}{9}mD^2 + \frac{8}{9}mD^2 = \frac{24}{9}mD^2 = 8/3\ mD^2$$

B. Moment of inertia fanned out

$I_2 = 3mD^2$.

C. initial angular momentum

$L_1 = I_1\omega_1 = (8/3\ mD^2)\omega_1$.

D. Final angular momentum

The important idea here is conservation of angular momentum. Final angular momentum is the same as initial angular momentum, $L_2 = L_1 = I_1\omega_1 = (8/3\ mD^2)\omega_1$.

E. Final angular velocity

Final angular momentum is final angular velocity times final moment of inertia: $L_2 = I_2\omega_2$. From conservation of angular momentum:

$$I_2\omega_2 = I_1\omega_1$$
$$\omega_2 = \omega_1 I_1/I_2 = \omega_1\frac{8/3mD^2}{3mD^2} = \omega_1\frac{8/3}{3}$$
$$\omega_2 = 8/9\ \omega_1$$

The angular velocity decreased as the moment of inertia increased.

F. Conservation of kinetic energy

The initial kinetic energy was $1/2\ I_1\ \omega_1^2$; the final kinetic energy was $1/2\ I_2\ \omega_2^2$.

$$K_1 = 1/2\ I_1\omega_1^2$$
$$\begin{aligned} K_2 &= 1/2\ I_2\omega_2^2 \\ &= 1/2\ I_2\ (\omega_1 I_1/I_2)^2 \\ &= 1/2\ (I_1^2/I_2)\omega_1^2 \\ &= 1/2\ I_1\omega_1^2(I_1/I_2) \\ K_2 &= K_1\ (I_1/I_2) \end{aligned}$$

We see that $K_1 \neq K_2$, Kinetic energy is absolutely not conserved; final kinetic energy is less than initial kinetic energy by the same factor that the initial moment of inertia is less than the final moment of inertia.

5. Hooke's law oscillator

In these questions, we consider the oscillations of two different masses on the same spring. The exam referred to them as m_1 and m_2, $m_2 > m_1$. I think it might be easier to follow if we say that $m_2 = \zeta m_1$, where ζ is a number greater than 1. So $m_1 = m$ and $m_2 = \zeta m$.

In both cases, the initial displacement of the spring is A and the initial speed is zero. The angular frequencies of oscillation will be $\omega_1^2 = k/m$ and $\omega_2^2 = k/\zeta m = \omega_1^2/\zeta$. In other words, $\omega_2 = \omega_1/\sqrt{\zeta}$.

A. Oscillation periods

The oscillation periods will be $T_1 = 2\pi/\omega_1$ and $T_2 = 2\pi/\omega_2 = \sqrt{\zeta}2\pi/\omega_1 = \sqrt{\zeta}T_1$. So the oscillation period is longer for the heavier mass.

B. Kinetic energy

The kinetic energy is entirely from work done by the spring. In both cases, the spring works over the same distance; the kinetic energy at any point is determined by the potential energy at that point. The mass does not affect this.

C. Maximum speed

The maximum speed of a harmonic oscillator is $A\omega$. The displacement amplitude A is the same for both oscillators here, but $\omega_2 = \omega_1/\sqrt{\zeta}$. Thus the maximum speeds are also related as $v_2 = v_1/\sqrt{\zeta}$. The heavier oscillator is slower at each point in its cycle.

D. Maximum net force

The spring force depends only on position. The maximum force occurs where the spring's elongation or compression equals the amplitude, which is the same for both oscillators.

E. Maximum acceleration

Acceleration is net force divided by mass: with the same net force, the lighter mass will accelerate more. In terms of the kinematic equations, the amplitude of acceleration is $A\omega^2$. Because $\omega_2^2 = \omega_1^2/\zeta$, the heavier mass has an acceleration that is a factor of ζ less than the lighter mass.

6. Simple Pendulum

The information we are given is an initial angle of 5°, a mass of 30 grams, and a length of 0.200 meters.

A. Oscillation amplitude

This is just a conversion problem. The initial angle is five degrees, and we want to find what that is in radians. A reasonable conversion equality is π rad = 180°, so $5° \cdot (\pi/180°) = \pi/36$ rad = 0.0873 rad.

B. Angular speed at the bottom

This question asked for the pendulum's angular speed, not the oscillator's angular frequency. The angular frequency is constant; the angular speed, which is the derivative of angular displacement with respect to time, varies sinusoidally.

The angular displacement is $\theta = \Theta \sin(\omega t)$, its derivative, the angular velocity, is $\omega\Theta\cos(\omega t)$. The speed is highest at the bottom of the pendulum's arc, so its value is the amplitude $\omega\Theta$. We know Θ is 0.0873 radians; ω is $\sqrt{g/L} = \sqrt{(9.8\ \text{m/s}^2)/(0.200\ \text{m})} = 7/\text{s}$.

Putting them together gives us $\omega\Theta = 0.6111$ rad/s.

You can also find the *tangential* speed by conservation of energy; the height difference between the top and bottom of the pendulum's arc is $h = L(1 - \cos\Theta)$, and the speed at the bottom is $\sqrt{2gh}$. Then you'll need to convert to angular speed by using v/L. You will get nearly the same answer.

Nearly, not exactly, because the small angle approximation is an approximation.

C. Kinetic energy at the bottom

The most straightforward way to find this is to convert the angular speed $\omega\Theta$ to tangential speed $v = \omega\Theta L = 0.12217$ m/s. Then $K = 1/2\ mv^2 = 2.24 \times 10^{-4}$ J.

It's about the same amount of work to find the rotational kinetic energy $1/2\ I(\omega\Theta)^2$. First you need to find $I = mL^2 = 0.0012\ \text{kg} \cdot \text{m}^2$, then you are on your way.

$K = 1/2\ (0.0012\ \text{kg} \cdot \text{m}^2)(0.6111\ \text{rad/s})^2 = 2.24 \times 10^{-4}$ J.

D. Time to return to release

The time to return to the release position is exactly the oscillation period $T = 2\pi/\omega$. This gives 0.896 seconds.

7. Physical pendulum

The torque due to gravity on the pendulum is $-MgL\sin\theta$. At small θ, this is approximately $-MgL\theta$. This is a torsional oscillator with torsional spring constant $\kappa = MgL$. Its oscillation angular frequency is $\omega = \sqrt{\kappa/I}$. where I is the moment of inertia. In a simple pendulum, $I = ML^2$, but if you can't neglect the center-of-mass moment of inertia I_{cm} of the swinging object, its moment of inertia is $I = I_{cm} + ML^2$. In this particular case, the center-of-mass moment of inertia of the square bob is $I_{cm} = \frac{1}{6}Ma^2$. Thus

$$\omega^2 = \frac{\kappa}{I} = \frac{MgL}{I_{cm} + ML^2} = \frac{MgL}{\frac{1}{6}Ma^2 + ML^2} = \frac{gL}{a^2/6 + L^2}$$

We are almost done. We just need to find $T = 2\pi/\omega$.

$$T = 2\pi\sqrt{\frac{a^2/6 + L^2}{gL}} = 2\pi\sqrt{\frac{a^2}{6gL} + L/g}$$

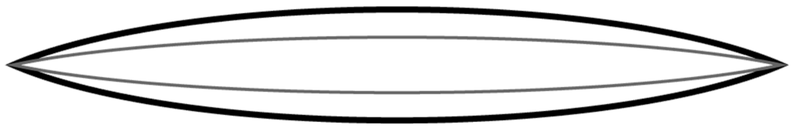
Note that this gives the simple pendulum formula if $a \ll L$.

8. Fundamental vibration of a guitar string

We are given the string's length $L = 65$ cm and fundamental frequency $f = 82.41$ Hz at the outset.

A. Shape of the fundamental vibration

In the fundamental vibration, there are no nodes between the clamped ends of the string.



B. Transverse or longitudinal

The motion of the string is transverse to the propagation of the wave.

C. Wavelength

Between the nodes at the ends of the string is half of a wavelength, so $\lambda = 2L = 130$ cm.

D. Propagation speed

The most straightforward way to find the propagation speed is $v = \lambda f = 107.1$ m/s.

E. Angular frequency

Angular frequency ω is related to frequency f by $\omega = 2\pi f$. This gives 518 rad/s.

F. Angular wavenumber

Angular wavenumber is related to wavelength as angular frequency is related to period: $k = 2\pi/\lambda$. Here this gives $k = 4.83$ rad/m.

G. Mass of the string

Here's where we use the string transverse wave speed formula $v = \sqrt{F/\mu}$. F is given as the tension of 65.0 newtons. The wave propagation speed v we found above. The length density μ is mass per unit length m/L. That's how we introduce the string's mass m into the formulas.

$$v = \sqrt{F/\mu}$$
$$v^2 = \frac{F}{m/L}$$
$$m = FL/v^2 = 0.00368 \text{ kg} = 3.68 \text{ g}$$